Universality of Random Graphs

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Abstract

We prove that asymptotically (as $n \to \infty$) almost all graphs with n vertices and $10dn^{2-\frac{1}{2d}} \log^{\frac{1}{d}} n$ edges are universal with respect to the family of all graphs with maximum degree bounded by d. Moreover, we provide a polynomial time, deterministic embedding algorithm to find a copy of each bounded degree graph in every graph satisfying some pseudo-random properties. We also prove a counterpart result for random bipartite graphs, where the threshold number of edges is even smaller but the embedding is randomized.

1 Introduction

Given graphs H and G, an *embedding* of H into G is an injective edge-preserving map $f\colon V(H) \to V(G)$, *i.e.*, for every $e = \{u, v\} \in E(H)$, we have $f(e) = \{f(u), f(v)\} \in E(G)$. We shall say that a graph H *is contained as a subgraph of* G if there is an embedding of H into G. Given a family of graphs $\mathcal{H}$, we say that G is universal with respect to $\mathcal{H}$, or $\mathcal{H}$*-universal*, if every $H \in \mathcal{H}$ is contained as a subgraph of G.

Consider the probability space of all graphs on n labelled vertices in which every pair of vertices forms an edge, randomly and independently, with probability p. We use the notation $G_{n,p}$ to denote a graph chosen randomly according to this probability measure; *i.e.*, for any graph G on n labelled vertices and with m edges, $\mathbb{P}[G_{n,p} = G] = p^m(1-p)^{\binom{n}{2}-m}$. We say that $G_{n,p}$ possesses a property Q asymptotically almost surely (**a.a.s.**) if $\mathbb{P}[G_{n,p} \in Q] = 1 - o(1)$.

The construction of sparse universal graphs for various families of graphs arises in the study of VLSI circuit design, and received a considerable amount of attention, see, *e.g.*, [1, 3, 4, 6, 8, 10] and their references. Since in some applications the cost of a vertex (site) may be higher than that of an edge (link), one is particularly interested in (almost) tight $\mathcal{H}$-universal graphs, *i.e.* graphs whose number of vertices is equal (or close) to $\max_{H\in\mathcal{H}} |V(H)|$.

In [6] it is proved that for all $\varepsilon > 0$ and $d > 0$ there exists $c > 0$ such that **a.a.s.** $G_{n,p}$, $p = c/n$, is $\mathcal{T}(d, (1-\varepsilon)n)$-universal, where $\mathcal{T}(d, (1-\varepsilon)n)$ is the family of trees with $(1-\varepsilon)n$ vertices and maximum degree at most d. In a related paper [11], the authors obtained an algorithm for finding bounded degree trees inside subgraphs of (n, d, λ)-graphs; in particular, the result of [6] is turned into an embedding algorithm. In this paper we study the universality of random graphs with respect to the family of all bounded degree graphs.

Let $d \in \mathbb{N}$ be a fixed constant and let $\mathcal{H}(n, d) = \{H \subseteq K_n : \Delta(H) \le d\}$ denote the class of (pairwise non-isomorphic) n-vertex graphs with maximum degree bounded by d and $\mathcal{H}(n, n; d) = \{H \subseteq K_{n,n} : \Delta(H) \le d\}$ be the corresponding class for balanced bipartite graphs.

By counting all unlabelled d-regular graphs on n vertices one can easily show that every $\mathcal{H}(n, d)$-universal graph must have

$$M = \Omega(n^{2-2/d}) \tag{1.1}$$

edges (see [3] for details). This lower bound was almost matched by a construction from [4], which was subsequently improved by similar constructions in [1] and [2], this last matching M up to a constant multiplicative factor. Those constructions were quite special and do not resemble a typical, or random, graph with the same number of edges. For that reason, in [3], we also studied the universality of random graphs.

For random graphs, slightly better lower bounds than (1.1) are known. Owing to the threshold for the property that every vertex belongs to a copy of K_{d+1} (see [13, Theorem 3.22 (i)]), the expected number of edges guaranteeing $\mathcal{H}(n, d)$-universality of $G_{n,p}$ must be at least $n^{2-2/(d+1)}(\log n)^{1/\binom{d+1}{2}}$, and, similarly, by [13, Theorem 4.9], it must be at least $n^{2-2/(d+1)}$ for $\mathcal{H}(n, d)$-universality of $G_{(1+\varepsilon)n,p}$. Similar bounds apply to the random bipartite graph $G_{n,n,p}$.

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In [3], it was proved that $G_{n,n,p}$ is **a.a.s.** $\mathcal{H}(n,n,d)$-universal if $p = cn^{-\frac{1}{2d}} \log^{\frac{1}{2d}} n$ and c is large enough, and that $G_{(1+\varepsilon)n,p}$ is **a.a.s.** $\mathcal{H}(n,d)$-universal if $p = cn^{-\frac{1}{d}} \log^{\frac{1}{d}} n$ if c is large enough. We summarize the best known results in Table 1.

In this paper we prove two related results. The first one significantly pushes down the edge density p guaranteeing the universality of $G_{n,n,p}$.

THEOREM 1.1. *Let $d \geq 1$ be fixed and let $p = C(d)\, n^{-\frac{1}{d}} \log^{\frac{1}{d}} n$, where $C(d)$ is a constant depending only on d. The random bipartite graph $G_{n,n;p}$ is, **a.a.s.**, $\mathcal{H}(n,n;d)$-universal.*

The second one, which we consider as the main result of this paper, on the cost of increase in p, establishes a tight universality of $G_{n,p}$ (and not of $G_{(1+\varepsilon)n,p}$), and provides, as opposed to Theorem 1.1, a constructive embedding.

THEOREM 1.2. *Let $d \geq 1$ be fixed and let $p = 20dn^{-\frac{1}{2d}} \log^{\frac{1}{d}} n$. The random graph $G_{n,p}$ is, **a.a.s.**, $\mathcal{H}(n,d)$-universal. Moreover, we can find a copy of each $H \in \mathcal{H}(n,d)$ in $G_{n,p}$ in deterministic polynomial time.*

It would be interesting to establish the actual thresholds for the $\mathcal{H}(n,n;d)$-universality of $G_{n,n,p}$ and the $\mathcal{H}(n,d)$-universality of $G_{n,p}$.

Let us emphasize here the algorithmic context of Theorem 1.2. The universality established in the results of [1, 3, 4] as well as in Theorem 1.1 is existential in the sense that the embedding was proved by probabilistic means. In Theorem 1.2, similarly to [2], a constructive and efficient embedding is provided.

The embedding algorithm in Theorem 1.2 is inspired by the algorithmic version of the *Blow-up Lemma* of Komlós, Sárközy, and Szemerédi [16]. In their setting, they essentially provided an algorithm to embed bounded degree spanning (bipartite) graphs into super-regular, dense, bipartite graphs. In our setting, we deal with sparse random graphs.

The algorithm works in two phases. It starts by embedding one vertex at a time until almost all of the vertices of the graph are embedded. The rest of the graph is embedded by finding a perfect matching in some auxiliary graph. The first phase is greedy (it never regrets a decision) but takes into consideration a few invariants that guarantee that the embedding of the whole graph can be done. This structure is quite similar to [16]. However, several differences and subtleties are inherent to the sparse random graph case.

2 Universality of random bipartite graphs

The proof of tight universality for random bipartite graphs uses the following strategy developed earlier in [5, 17, 18]. Let $G = (U, W; E) \subseteq K_{n,n}$ be some fixed graph. We partition $W = W_1 \cup \cdots \cup W_{d^2}$ where $|W_i| = m = n/d^2$. Given any $H = (X, Y, E_H) \in \mathcal{H}(n,n;d)$ we may apply the Hajnal–Szemerdi Theorem to the graph $H^2[Y] = (Y, \{\{y_1, y_2\} : \mathrm{dist}(y_1, y_2) = 2\})$. Since the maximum degree in $H^2[Y]$ is at most $d(d-1) \leq d^2 - 1$, this theorem can be used to partition the set Y into equal-sized sets $Y_1, \ldots, Y_{d^2}$ with each Y_i being an independent set in $H^2[Y]$. By construction, every Y_i is a *two-independent* set over H, meaning that every two vertices in Y_i are at distance at least 3 from each other.

Without loss of generality we may and will assume that each vertex $y \in Y$ has degree $\deg_H(y) = d$ (allowing the vertices of X to have higher degrees). This can be achieved by adding extra edges, if necessary, while making sure that the sets Y_i remain two-independent. This is possible, since $md = n/d \leq n/2$.

We shall construct an embedding $f\colon V(H) \to V(G)$ of H into G such that $f(X) = U$ and $f(Y_i) = W_i$ for every i. First, we take a random bijection $\pi\colon X \to U$. Then, we show that, **a.a.s.**, one can construct bijections $f_i\colon Y_i \to W_i$ such that f defined by $f|_{Y_i} \equiv f_i$ and $f|_X \equiv \pi$ is a valid embedding of H into G.

Given π and i, let $A = A(i,\pi)$ be an auxiliary bipartite graph between vertex sets Y_i and W_i such that $\{y, w\}$ is an edge of A iff $\Gamma_G(w) \supseteq \pi(\Gamma_H(y))$. Our goal is to show that for almost all bijections π and for all $i = 1, \ldots, d^2$, the graph $A(i,\pi)$ has a perfect matching M_i. These perfect matchings naturally define bijections f_i as required.

We shall prove that a graph satisfying the following pseudo-random properties is $\mathcal{H}(n,n;d)$-universal.

P1(ν) For all $i = 1, \ldots, d^2$ and for every collection $\mathcal{S}$ of $s \leq (1-\nu)m$ pairwise disjoint, non-empty subsets of U, each of size d, and for every subset $T \subseteq W_i$ of $t = |T| = m - s + 1$ vertices, there exist $w \in T$ and $S \in \mathcal{S}$ such that $\Gamma_G(w) \supseteq S$.

P2 For all $w, w' \in W$, $w \neq w'$, we have

$$\deg(w) \sim np \qquad \text{and}$$
$$\deg(w,w') = |\Gamma(w) \cap \Gamma(w')| \sim np^2.$$

P3 Given any $U' \subseteq U$ with $|U'| \geq n/2$ there are at most $100/p$ vertices $w \in W$ such that $|\Gamma_G(w) \cap U'| \leq pn/4$.

Property **P1**(ν) is readily established for random graphs with the help of Lemma 2.1. The other properties follow from direct applications of Chernoff and union bounds.

Table 1: Summary of (best) known universality results (log powers are omitted).

	Universality of	Upper bound	Lower bound	Reference
Random Graphs	$\mathcal{H}(n,n;d)$ in $G_{n,n;p}$	$p = n^{-\frac{1}{d}}$	$n^{-2/(d+1)}$	Theorem 1.1
	$\mathcal{H}(n,d)$ in $G_{n,p}$	$p = n^{-\frac{1}{2d}}$		Theorem 1.2
	$\mathcal{H}(n,d)$ in $G_{(1+\varepsilon)n,p}$	$p = n^{-\frac{1}{d}}$		[3]
Constructive	$\mathcal{H}(n,d)$ in G	$\lvert E(G)\rvert = n^{2-2/d}$	$n^{2-2/d}$	[1, 2, 4]

LEMMA 2.1. *Fix $k \geq 1$ and let $N = N(n)$ and $m = m(n) \leq N/k$. For all $\nu > 0$ there exist $c > 0$ such that if $p \geq cn^{-1/k}(\log n)^{1/k}$ then the random bipartite graph $G(U, W; p)$, where $|U| = N$ and $|W| = m$, has the following property* **a.a.s.**:

For every collection $\mathcal{S}$ of $s \leq (1-\nu)m$ pairwise disjoint, non-empty subsets of U, each of size at most k, and for every subset $T \subseteq W$ of $t = |T| = m - s + 1$ vertices, there exist $w \in T$ and $S \in \mathcal{S}$ such that $\Gamma_G(w) \supseteq S$.

To prove the existence of a perfect matching in A, we will use Hall's condition. First note that by Property $\mathbf{P1}(\nu)$, for *all* bijections π, Hall's condition holds for every $S \subseteq Y_i$ of size $|S| = s \leq m - \nu m$. To cover the remaining cases, we will show that for almost all π, for every $T \subseteq W_i$ of cardinality $1 \leq t = |T| \leq \nu m$, we have $|\Gamma_A(T)| \geq t$. This establishes Hall's condition for all $S \subseteq Y_i$, and consequently the existence of a perfect matching in A follows.

The proof will be split in two cases: (1) small t, meaning $t \leq \alpha/p$, for some $\alpha = \alpha(d)$ and (2) large t, meaning $t > \alpha/p$. The probability space over bijections π is seen by different perspectives in those two cases. Notice that we only have to prove that, for any fixed $i \in [d^2]$, **a.a.s.**, there is a matching in $A = A(i, \pi)$; the union bound then gives us that the same holds for all i simultaneously.

LEMMA 2.2. *There exists $\alpha = \alpha(d)$ such that,* **a.a.s.**, *for every $T \subseteq W_i$ of cardinality $t \leq \alpha/p$, we have $|\Gamma_A(T)| \geq tp^d n/(2^{2d+3}d^2)$.*

Proof. [Proof outline] First, we pick representative sets $N_k \subseteq \Gamma_G(w_k)$ for each $w_k \in T = \{w_1, \ldots, w_t\}$ which are all disjoint and have cardinality $pn/2$. The permutation π is then exposed by steps: on the kth step, we expose π^{-1} over the set N_k. Observe that if $\Gamma_H(y) \subseteq \pi^{-1}(N_k)$ for some $y \in Y_i$ then (y, w_k) is an edge of A. Also notice that $\pi^{-1}(N_k)$ is a uniformly sampled $(pn/2)$-set from $X \setminus \pi^{-1}(\bigcup_{j<k} N_j)$.

Since t is small, this procedure does not expose a large part of π and at each step there are many $y \in Y_i$ that may be captured by the uniformly sampled set $\pi^{-1}(N_k)$.

LEMMA 2.3. *For the same α of Lemma 2.2 and a sufficiently small $\nu = \nu(d)$ we have,* **a.a.s.**, *for every $T \subseteq W_i$ of cardinality $\alpha/p < t \leq \nu m$, $|\Gamma_A(T)| \geq t$.*

Proof. [Proof outline] Assume that $Y_i = \{y_1, \ldots, y_m\}$. We expose the random permutation π by steps: on the kth step, we expose the values of π over the set $X_k = \Gamma_H(y_k)$. The set $\pi(X_k)$ is a uniformly sampled d-set from $U_k = U \setminus \pi(\bigcup_{j<k} X_j)$.

Since U_k is large for all k, most vertices of T have large degree inside U_k. Using the Bonferroni inequality we may estimate how many d-sets are contained in some neighborhood $\Gamma_G(w) \cap U_k$, with $w \in T$. The probability of choosing $\pi(X_k)$ inside some neighborhood can be lower bounded by such an argument and this yields a proof of the lemma.

Lemmas 2.2 and 2.3 and the union bound shows that for any $H \in \mathcal{H}(n,n;d)$ and G satisfying pseudo-random properties $\mathbf{P1}(\nu)$ (with ν as in Lemma 2.3), $\mathbf{P2}$ and $\mathbf{P3}$, **a.a.s.**, a random permutation π ensures that $A(i, \pi)$ has a perfect matching for all i, hence, we can define all $f_i \colon Y_i \to W_i$ from those matchings and establish the embedding needed to conclude the proof of Theorem 1.1.

3 An embedding algorithm for bounded degree graphs

For this section, let $d \in \mathbb{N}$ and $\varepsilon = \{3(d^2+1)\}^{-1}$ be fixed and let $p = C\,n^{-1/(2d)} \log^{1/d} n$, where $C = 20d$. We shall assume that n is large enough with respect to d and ε.

We start by defining a suitable notation for the common neighborhood of a vertex-set.

DEFINITION 3.1. *Given a graph G and $S \subseteq V(G)$, let*

$$\Gamma_G^{\cap}(S) = \{x \in V(G) \,:\, S \subseteq \Gamma_G(x)\},$$

where $\Gamma_G(x)$ denotes, as usual, the neighborhood of the vertex x in G.

Assume that $V(H) = [n]$ and that the last $n/(d^2+1)$ vertices are at distance at least 3 from each other.

REMARK 3.1. *Notice that we can obtain a set of $n/(d^2+1)$ vertices that are at distance at least 3 from each other by a greedy method: start with an arbitrary vertex x, eliminate at most d^2 vertices that might be at distance 1 or 2 from x, select another arbitrary vertex and repeat.*

Now let us describe an algorithm that tries to embed any graph H with n vertices and $\Delta(H) \leq d$ into some other n-vertex graph G (hence, if it succeeds, the embedding will be a bijection). We use a sequence of auxiliary bipartite graphs $I_0 = K_{V(H),V(G)}, I_1, \dots$ with classes contained in $V(H)$ and $V(G)$ respectively. Let f_j denote the embedding constructed after iteration j. The vertices of I_j are the non-embedded vertices of H and the free vertices of G. The edges in the graph I_j, $j \geq 1$, indicate the possible extensions of f_{j-1}. In particular, for every non-embedded $x \in V(H)$, $\Gamma_j(x) = \Gamma_{I_j}(x) \subseteq V(G)$ is given by[1]

$$\Gamma_G^{\cap}(f_j(\Gamma_H(x))) \setminus f_j(V(H)). \tag{3.2}$$

The set $\Gamma_j(x)$ can be thought as a *candidate* set for x, since x could be mapped to any element of $\Gamma_j(x)$ while still preserving the embedding.

The algorithm operates in two phases. In the first phase, we attempt to embed between $(1 - 2\varepsilon)n$ and $(1 - \varepsilon)n$ vertices, one by one. By our choice of ε, the vertices left to be embedded in the second phase (if the first phase is successful) are such that the distance between any two of them is at least 3, that is, their neighborhoods are disjoint and they form an independent set. This condition is enough to ensure that if we embed one vertex of such a set, all the other non-embedded vertices are only affected by possibly missing one candidate. This means that the second phase consists solely of finding a perfect matching in the remaining graph I_t, where t is the last step of the first phase.

Thus, phase one deserves a more delicate analysis. We shall give a brief informal view first. There are a few local conditions that the algorithm tries to preserve: the degree of every vertex $x \in V(H)$ in the sequence of graphs I_j is tightly related to the number of embedded neighbors of x at the jth step; every vertex $w \in V(G)$ has a reasonable number of unused neighbors in G; the degree of $w \in V(G)$ in I_j is lower bounded.

[1] We shall abuse notation and consider $f_j(S)$ as being the image of the intersection of S and the domain of f_j (which is a subset of $V(H)$).

Let us describe how the graph I_j changes when the embedding is extended by one vertex, say $x \mapsto w$. On the $V(G)$ side of I_j these changes are mild. Indeed, any unused vertex $w' \in V(G)$ may only lose vertices in $\Gamma_H(x) \cup \{x\}$. On the $V(H)$ side, we may have more drastic changes. If $x' \in V(H)$ is not embedded then it may lose w as a candidate, and, if in addition $x' \in \Gamma_H(x)$, then it loses considerably more vertices. In this case, the new candidate set of x' consists of the old one intersected with $\Gamma_G(w)$.

In order to keep the good local conditions mentioned above, we might have to deal with vertices that fail those conditions. More concretely, we shall define invariants for the sequence I_j. Assuming that the local conditions are satisfied for I_j, we must extend the embedding and produce an I_{j+1} that corresponds to the extended embedding such that I_{j+1} also respects all the local conditions. The way phase one accomplishes this is by dealing with vertices that fail any of those conditions right away: if the current embedding makes some *bad* vertex fail a local condition, the algorithm picks a bad vertex w—we then call w a *critical* vertex—and extends the embedding using w. The potential problem here is that we might have to deal with more and more vertices, eventually stumbling upon a vertex that simply cannot be used.

We remark that not all bad vertices are necessarily critical. A vertex $x \in V(H)$ might become bad if it does not have a sufficient number of candidates, but this number depends on how many neighbors of x are already embedded. If some $x' \in \Gamma_H(x)$ becomes critical before x it may save x from its bad reputation.

By imposing a few structural conditions on a deterministic graph G, we are able to prove that the set of all critical vertices has a very limited size. In fact, so limited that it is negligible compared to the degrees in I_j.

Let $F_j(x)$ be the set of non-embedded neighbors of $x \in V(H)$, that is, $\Gamma_H(x) \setminus f_j^{-1}(V(G))$. Let $\nu_j(x) = |f_j^{-1}(V(G)) \cap \Gamma_H(x)|$ be the number of embedded neighbors of $x \in V(H)$. Both $F_j(x)$ and $\nu_j(x)$ depend on f_j, hence, when f_j is extended by possibly several vertices, these values change accordingly. It will be convenient to identify f_j by a matching in $V(H) \times V(G)$.

At the beginning of the jth iteration (line 1.5), the following invariants hold (f_j is determined by the matching M).

INVARIANT 3.1. *For all non-embedded $x \in V(H)$, we have*

$$\deg_j(x) = |\Gamma_j(x)| \geq (p/4)^{\nu_j(x)}\varepsilon n.$$

Algorithm 1: Phase 1

1.1 $M \leftarrow \emptyset$; // initialize embedding
1.2 $V_H \leftarrow V(H)$, $V_G \leftarrow V(G)$; // set of free vertices in G and H
1.3 $I_0 \leftarrow K_{V_H,V_G}$, $j \leftarrow 0$; // initialize auxiliary graph
1.4 **while** $|M| < (1-2\varepsilon)n$ **do**
1.5 $x \leftarrow \min(V_H)$;
1.6 $\Gamma_j(x) \leftarrow$ **clean-up**(x);
1.7   **if** $\Gamma_j(x) = \emptyset$ **then**
1.8   **abort**; // could not embed vertex x
1.9 $y \leftarrow \min(\Gamma_j(x))$;
1.10   **extend-embedding**(x, y);
1.11   **restore-invariants**;
1.12 $I_{j+1} \leftarrow I_j$, $j \leftarrow j+1$;

Procedure restore-invariants

2.1 **while** $\exists x$ *failing Invariant 3.1 or* $\exists w$ *failing Invariant 3.2* **do**
2.2 $\mathtt{cr} \leftarrow \mathtt{cr} + 1$;
2.3   **if** $\mathtt{cr} \geq \mathtt{cr}^{\max}$ **then**
2.4   **abort** phase 1; // too many critical vertices!
2.5   **if** $\exists x$ *failing Invariant 3.1* **then**
2.6 $\Gamma_j(x) \leftarrow$ **clean-up**(x);
2.7   **if** $\Gamma_j(x) = \emptyset$ **then**
2.8   **abort** phase 1; // failed on critical vertex of H
2.9 $w \leftarrow \min(\Gamma_j(x))$;
2.10   **extend-embedding**(x, w);
2.11   **else if** $\exists w$ *failing Invariant 3.2* **then**
2.12 $Z \leftarrow \{x \in \Gamma_j(w) \, : \, w \in$ **clean-up**$(x)\}$;
2.13   **if** $Z = \emptyset$ **then**
2.14   **abort** phase 1; // failed on critical vertex of G
2.15 $x \leftarrow \min(Z)$;
2.16   **extend-embedding**(x, w);

INVARIANT 3.2. *For all* $w \in V(G) \setminus f_j(V(H))$,

$$i. \ \deg_j(w) \geq \frac{1}{2}\sqrt{n}\log n \quad and$$

$$ii. \ |\Gamma_G(w) \setminus f_j(V(H))| \geq \frac{1}{2}\varepsilon p n.$$

We shall also have an inner invariant. Namely, an invariant for Procedure **restore-invariants**. First, let us fix the maximum number of critical vertices that the algorithm may handle, $\mathtt{cr}^{\max} = 2(d^2+2)\sqrt{n}$.

INVARIANT 3.3. *At the beginning of every iteration of Procedure* **restore-invariants** *(line 2.2), we have, for every non-embedded* $x \in V(H)$,

$$\deg_j(x) \geq (p/4)^{\nu_j(x)}\varepsilon n - \mathtt{cr}^{\max}.$$

Procedure clean-up(x)

3.1 $F_j(x) \leftarrow \Gamma_H(x) \cap V_H$; // non-embedded neighbors of x
3.2 **foreach** $y \in F_j(x)$ **do**
3.3 $B_y \leftarrow \{z \in \Gamma_j(x) \, : \, |\Gamma_G(z) \cap \Gamma_j(y)| < (p/4)^{\nu_j(y)+1}\varepsilon n\}$;
3.4 **return** $\Gamma_j(x) \setminus \bigcup_{y \in F_j(x)} B_y$;

Procedure extend-embedding(u, v)

4.1 $M \leftarrow M \cup \{(u,v)\}$;
4.2 $V_H \leftarrow V_H \setminus \{u\}$;
4.3 $V_G \leftarrow V_G \setminus \{v\}$;
4.4 $I_j \leftarrow (V_H, V_G;$ edges determined by eq. (3.2));

4 Universality of random graphs

In this section we show that both phases of the algorithm succeed in obtaining the desired embedding as long as G is a graph satisfying the following:

(i) for every pair of (not necessarily disjoint) sets $A, B \subseteq V(G)$ such that $|A| \cdot |B| \geq n \log n / p$, we have
$$e_G(A,B) \geq \frac{2}{3}p\,|A|\,|B|,$$
where edges in $A \cap B$ are counted twice;

(ii) $\delta(G) \geq 0.9pn$;

(iii) for every set $T \subseteq V(G)$ of size $|T| \geq \varepsilon p n/4$ and every collection of $k = C_2 p^{-d} \log n$ disjoint sets $X_1, \ldots, X_k \subseteq V(G) \setminus T$ having at most d elements each, there is some $i \in [k]$ such that
$$|\Gamma_G^{\cap}(X_i) \cap T| \geq \frac{1}{2}p^{|X_i|}\,|T|;$$

(iv) for every $T \subseteq V(G)$ with $|T| = \sqrt{n}$ and disjoint sets $X_1, \ldots, X_a \subseteq V(G) \setminus T$, with $a > \varepsilon n/(d^2+1)$, having at most d elements each, there is $y \in T$ such that $X_i \subseteq \Gamma_G(y)$ holds for at least $\frac{1}{2}\sqrt{n}\log n$ indices $i \in [a]$;

(v) for every $T \subseteq V(G)$ with $|T| = r = \frac{1}{2}\sqrt{n}\log n$ and every collection of disjoint sets $X_1, \ldots, X_r \subseteq V(G) \setminus T$ having at most d elements each, there is $w \in T$ and $i \in [r]$ such that $X_i \subseteq \Gamma_G(w)$.

Standard use of Chernoff bounds prove that **a.a.s.** $G_{n,p}$ satisfies all these properties.

THEOREM 4.1. *Properties (i–v) are* **a.a.s.** *satisfied by* $G_{n,p}$.

4.1 Phase one succeeds In this subsection we deal with the analysis of the first phase of the embedding algorithm.

THEOREM 4.2. *Assume that G is a graph satisfying (i–v) and H has degree bounded by d. Both graphs have n vertices, with n sufficiently large. Then, Algorithm 1 does not abort.*

If the algorithm does not abort, it finds a valid (partial) embedding of H into G. Furthermore, the only vertices that remain unembedded after the execution are at distance at least 3 from each other in H. Therefore, one can proceed to phase two in order to complete the embedding since any perfect matching in the remaining auxiliary graph is a valid extension of the embedding.

The proof of Theorem 4.2 consists of proving that all invariants are kept (in particular, Procedure **restore-invariants** is correct) and that none of the abort conditions ever hold (see lines 1.8, 2.4, 2.8, 2.14).

4.2 Phase two succeeds After phase one succeeds, we are left with independent vertices with disjoint neighborhoods to embed. Indeed, phase one embeds at least $(1-2\varepsilon)n$ vertices and at most $\mathrm{cr}^{\max}$ of them are critical. Thus, by our choice of ε, the unembedded vertices must belong to the $n/(d^2+1)$ last vertices of $V(H)$.

Furthermore, by the Invariants 3.1 and 3.2.i, we have minimum degree on both sides of I_t, where t is the last iteration of the first phase. Let A denote the set of unembedded vertices of H and B the set of free vertices of G. Clearly $|A| = |B|$, furthermore, by our choice of C, we can ensure that the minimum degree on both sides is at least $\frac{1}{2}\sqrt{n}\log n$. Hence, if Hall's condition is not satisfied in I_t, there must be a set $A' \subseteq A$ such that $|\Gamma_t(A')| < |A'|$. Setting $B' = B \setminus \Gamma_t(A')$, we must have $|\Gamma_t(B')| < |B'|$. It follows that $|A'|, |B'| \geq \frac{1}{2}\sqrt{n}\log n$. Furthermore, there is no edge in I_t between A' and B'. But this contradicts Property (v).

Algorithm 5: Find Embedding

5.1 execute Phase 1;
5.2 $M' \leftarrow$ perfect matching in I_j; // Phase 2
5.3 **return** $M \cup M'$;

COROLLARY 4.1. *Given a graph G on n vertices satisfying (i–v) and a graph H on n vertices such that $\Delta(H) \leq d$, there is a polynomial-time algorithm that finds an embedding $H \hookrightarrow G$.*

From Theorems 4.1 and Corollary 4.1 we obtain our universality result for random graphs, Theorem 1.2.

References

[1] N. Alon and M. Capalbo, Sparse universal graphs for bounded-degree graphs, *Random Structures Algorithms*, to appear, 11pp., published on-line in Wiley InterScience (www.interscience.wiley.com), DOI 10.1002/rsa.20143.

[2] N. Alon and M. Capalbo, Optimal universal graphs with deterministic embedding, submitted.

[3] N. Alon, M. Capalbo, Y. Kohayakawa V. Rödl, A. Ruciński, and E. Szemerédi, Universality and tolerance (Extended Abstract) *In Proceedings of the 41st IEEE Annual Symposium on FOCS*, pages 14-21.

[4] N. Alon, M. Capalbo, Y. Kohayakawa V. Rödl, A. Ruciński, and E. Szemerédi, Near-optimum universal graphs for graphs with bounded degrees (Extended Abstract) *APPROX-RANDOM 2001, LNCS 3139* (2001) 170-180.

[5] N. Alon and Z. Füredi, Spanning subgraphs of random graphs, *Graphs and Combinatorics* 8 (1992), 91-94.

[6] N. Alon, M. Krivelevich and B. Sudakov, Embedding nearly spanning bounded degree trees, Combinatorica, to appear

[7] N. Alon and R. Yuster, Threshold functions for H-factors, *Combinatorics, Probability and Computing* 2 (1993), 137-144.

[8] S. N. Bhatt, F. Chung, F. T. Leighton and A. Rosenberg, Universal graphs for bounded-degree trees and planar graphs, *SIAM J. Disc. Math.* 2 (1989), 145-155.

[9] Béla Bollobás. *Random Graphs.* Cambridge University Press, 2nd edition, 2001.

[10] M. R. Capalbo and S. R. Kosaraju, Small universal graphs, Proc. STOC (1999), 741–749.

[11] D. Dellamonica Jr. and Y. Kohayakawa. An algorithmic Friedman–Pippenger theorem on tree embeddings and applications to routing (extended abstract). In

SODA 2006: Proceedings of the 17th annual ACM-SIAM symposium on Discrete algorithms. Society for Industrial and Applied Mathematics, 2006.

[12] A. Hajnal and E. Szemerédi, Proof of a conjecture of Erdös, in: *Combinatorial Theory and its Applications*, Vol. II (P. Erdös, A. Renyi and V. T. Sós eds.), Colloq. Math. Soc. J. Bolyai 4, North Holland, Amsterdam 1970, 601-623.

[13] S. Janson, T. Łuczak and A. Ruciński, *Random Graphs*, Wiley, New York, 2000.

[14] Y. Kohayakawa and V. Rödl. Regular pairs in sparse random graphs. I. *Random Structures Algorithms*, 22(4):359–434, 2003.

[15] J. Komlós, G. Sárközy and E. Szemerédi, The Blow-up Lemma, *Combinatorica* 17 (1997) 109-123

[16] János Komlós, Gabor N. Sárközy, and Endre Szemerédi. An algorithmic version of the blow-up lemma. *Random Structures Algorithms*, 12(3):297–312, 1998.

[17] V. Rödl, A. Ruciński, Perfect matching in ϵ-regular graphs and the Blow-up Lemma, *Combinatorica* 19 (3) (1999) 437-452.

[18] A. Ruciński, Matching and covering the vertices of a random graph by copies of a given graph, *Discrete Math.* 105 (1992), 185-197.