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COVERING THE EDGES OF A RANDOM HYPERGRAPH BY CLIQUES

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Abstract

We determine the order of magnitude of the minimum clique cover of the edges of a binomial, r-uniform, random hypergraph $G^{(r)}(n,p)$, p fixed. In doing so, we combine the ideas from the proofs of the graph case ($r = 2$) in Frieze and Reed [*Covering the edges of a random graph by cliques*, Combinatorica 15 (1995) 489–497] and Guo, Patten, Warnke [*Prague dimension of random graphs*, manuscript submitted for publication].

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1. Introduction

For an r-uniform hypergraph (briefly, an r-graph) $H = (V, E)$ and a set S, a *representation of H on S* is an assignment of subsets $S_v \subset S$, $v \in V$, in such a way that for each $R \in \binom{V}{r}$ we have $\bigcap_{v\in R} S_v \neq \emptyset$ if and only if $R \in E$. To observe that any r-graph admits such a representation, assign to each vertex v the set $\{e : v \in e \in E\}$ of all edges e containing v. Then, $\{v_1, \ldots, v_r\} \in E$ if and only if $\bigcap_{j+1}^{r} S_{v_j} \neq \emptyset$.

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Definition. The *representation number* $\theta_1(H)$ of H is the smallest cardinality of a set S which admits a representation of H. Equivalently, $\theta_1(H)$ is also the smallest number of cliques needed to cover all edges of H (see Appendix for a proof of the equivalence).

It is perhaps interesting to note (cf. [6]) that the maximum of $\theta_1(H)$ over all r-graphs H on n vertices equals the Turán number for the r-uniform clique $K_{r+1}^{(r)}$ on $r+1$ vertices which is unknown even for $r=3$.

We determine a typical order of magnitude of the parameter θ_1 for a class of large *random* r-graphs. Given integers $n, r \geq 2$, and a real $0 < p < 1$, let $G^{(r)}(n,p)$ denote the random r-graph obtained by independent inclusion of each r-set with probability p. In particular, the number of edges of $G^{(r)}(n,p)$ is binomially distributed with expectation $\binom{n}{r}p$. We say that a property of r-sets $\mathcal{P}$ holds *asymptotically almost surely*, abbreviated to *a.a.s.*, if the probability $\mathrm{Prob}(G^{(r)}(n,p) \in \mathcal{P}) \to 1$ as $n \to \infty$. Throughout the paper p remains independent of n, while all logarithms are natural and denoted by log.

Theorem 1. *For every integer $r \geq 2$ and a constant $0 < p < 1$, there exist positive constants c_1 and c_2 such that a.a.s.*

$$c_1 \frac{n^r}{(\log n)^{r/(r-1)}} \leq \theta_1(G^{(r)}(n,p)) \leq c_2 \frac{n^r}{(\log n)^{r/(r-1)}}.$$

The case $r=2$ was proved by Frieze and Reed [3], and, in a stronger form, by Guo, Patten, and Warnke [4]. Here we follow the ideas from there, some of which have already originated in a paper by Alon, Kim, and Spencer [1]. The lower bound follows immediately from the upper bound on the order of the largest clique in $G^{(r)}(n,p)$ (see below). Hence, in the remaining sections we focus on the upper bound only.

***Proof of the lower bound in Theorem* 1.** Recall that p and r are constants independent of n. The expected number of cliques of order

$$t := \left\lceil \left(\frac{r!}{\log(1/p)} \log n \right)^{1/(r-1)} \right\rceil + r$$

in $G^{(r)}(n,p)$ is

$$\binom{n}{t} p^{\binom{t}{r}} \leq \left(\frac{en}{t} p^{(t-r)^{r-1}/r!} \right)^t \leq \left(\frac{en}{t} e^{-\log n} \right)^t = o(1)$$

as $n \to \infty$. Hence, a.a.s., there are no cliques of order t (or higher). On the other hand, by Chebyshev's inequality, there are, a.a.s., at least $\frac{1}{2}p\binom{n}{r}$ edges in

$G^{(r)}(n,p)$. Therefore, a.a.s., one needs at least

$$\frac{\frac{1}{2}p\binom{n}{r}}{\binom{t}{r}} \geq \frac{1}{2}p\left(\frac{n}{t}\right)^r = \Omega\left(\frac{n^r}{(\log n)^{r/(r-1)}}\right)$$

cliques to cover all edges of $G^{(r)}(n,p)$. ■

For the proof of the upper bound we now define a crucial notion. Let an r-graph G on a vertex set V, two integers $0 \leq s < j \leq |V|$, and a set $S \in \binom{V}{s}$ be given. A subset $J \subset V$ is called an (S,j)-*clique in* G if

- $J \supset S$, $|J| = j$, and
- $E_J := \{f \in \binom{J}{r} : f \cap (J \setminus S) \neq \emptyset\} \subset E(G)$, that is, $E(G)$ contains all r-element subsets of J except those which are subsets of S.

Note that $|E_J| = \binom{j}{r} - \binom{s}{r}$. Moreover, for $s = 0$, an $(\emptyset, j)$-clique is just any copy of the clique $K_j^{(r)}$ in G, while for $j < r$ every $J \in \binom{V}{j}$, $J \supseteq S$, is a (trivial) (S,j)-clique (with $E_J = \emptyset$).

2. AN EXPANDING PROPERTY OF RANDOM r-GRAPHS

Throughout the paper V is an n-vertex set of the random r-graph $G^{(r)}(n,p)$. Given $s \geq r-1$, and a set $S \in \binom{V}{s}$, let $X(S)$ be the number of $(S, s+1)$-cliques in $G^{(r)}(n,p)$. In other words, $X(S)$ counts the common neighbors of all $(r-1)$-element subsets of S. Clearly,

$$\mathbf{E}(X(S)) = (n-s)p^{\binom{s}{r-1}} := \mu(s). \tag{1}$$

The next lemma asserts that, for a wide range of s and for every s-element set S of vertices there are roughly the same number of $(S, s+1)$-cliques in $G^{(r)}(n,p)$. Let

$$k = \left\lfloor (\alpha \log n)^{1/(r-1)} \right\rfloor$$

for sufficiently small $\alpha > 0$.

Claim 2. *Let $\mathcal{A}$ be the event that for all $r-1 \leq s \leq k-1$ and all $S \in \binom{V}{s}$*

$$|X(S) - \mathbf{E}(X(S))| \leq n^{-1/3}\mathbf{E}(X(S)).$$

Then, $\text{Prob}(\mathcal{A}) = 1 - o(1)$.

Proof. Note that $\mu(s)$ is a decreasing function of s and, by the definition of k above, $\binom{k-1}{r-1} \leq (k-1)^{r-1} \leq \alpha \log n$. Thus, for large n,

$$\mu(s) \geq \mu(k-1) \geq (n/2)p^{\binom{k-1}{r-1}} \geq (n/2)p^{\alpha \log n} \geq \frac{1}{2}n^{0.99}, \tag{2}$$

if only $\alpha \log(1/p) \le 0.01$.

Recall that, for every $S \in \binom{V}{s}$, the random variable $X(S)$ counts the number of $(S, s+1)$-cliques and so, it is binomially distributed with expectation given by (1). Thus, by Chernoff's bound (see, e.g., [5], Corollary 2.3, Ineq. (2.9)), assuming n is sufficiently large,

$$\mathrm{Prob}(|X(S) - \mu(s)| > n^{-1/3}\mu(s)) \le 2\exp\{-n^{-2/3}\mu(s)/3\} \le \exp\{-n^{1/4}\},$$

where the last inequality follows from (2). Finally, by the union bound, summing over all choices of s and S,

$$\mathrm{Prob}(\neg\mathcal{A}) \le kn^k \exp\{-n^{1/4}\} = o(1). \qquad \blacksquare$$

3. Proof of Theorem 1

The upper bound in Theorem 1 will be a consequence of Claim 2 given in Section 2 and Lemma 4 to be stated below.

3.1. Notation

Before stating Lemma 4, we introduce a few parameters used therein. Very roughly, the lemma will claim the existence of a sequence of r-graphs $G_1, \dots, G_{i_0}$,

$$i_0 = \left\lceil \frac{r+1}{r-1} \log\log n \right\rceil,$$

which begins with the random r-graph $G_1 := G^{(r)}(n,p)$ and maintains throughout certain properties. As the proof of Lemma 4 will reveal, each next graph G_{i+1} will be derived from G_i by a random deletion of cliques of order k_i, where, for sufficiently small $\alpha > 0$,

$$k_i = \left\lfloor \left(\frac{\alpha \log n}{i} \right)^{1/(r-1)} \right\rfloor.$$

(Note that $k_1 = k$ defined earlier.)

In addition, some random edges of G_i will be deleted as well. The random procedure will be designed in such a way that the graphs G_i will shrink at the rate of $1/e$, thus resembling random r-graphs $G^{(r)}(n, p_i)$, where

$$p_i = pe^{1-i}.$$

The resemblance will be manifested by the behavior of the number of (S, j)-cliques in G_i, which, for all $0 \le s < j \le k_i$, will be close to the quantity

$$\mu_i(s,j) = \binom{n-s}{j-s} p_i^{\binom{j}{r} - \binom{s}{r}}. \tag{3}$$

Note that $\mu_i(s,j)$ is the expected number of (S,j)-cliques in a random r-graph $G^{(r)}(n,p_i)$.

In particular, for $s \ge r-1$, the quantity $\mu_1(s,s+1)=\mu(s)$ has been defined in (1). Note also that

$$\frac{\mu_{i+1}(s,j)}{\mu_i(s,j)} = (1/e)^{\binom{j}{r}-\binom{s}{r}}. \tag{4}$$

Let us now prove some bounds on $\mu_i(s,j)$.

Claim 3. *For all $1 \le i \le i_0$, all $0 \le s < j \le k_i$, sufficiently small α and sufficiently large n,*

(a) $\frac{\mu_i(s+1,j)}{\mu_i(s,j)} \le n^{-0.99}$;

(b) $\mu_i(s,j) \ge n^{0.99}$.

Proof. (a) Since $s < k_i \le \left(\frac{\alpha \log n}{i}\right)^{1/(r-1)}$, we have $is^{r-1} < \alpha \log n$. Hence, for sufficiently small α and large n,

$$\frac{\mu_i(s+1,j)}{\mu_i(s,j)} = \frac{j-s}{n-s} p_i^{-\binom{s}{r-1}} < \frac{j}{n}(e^i/p)^{s^{r-1}} \le \frac{j}{n}(e/p)^{is^{r-1}} < \frac{j}{n}(e/p)^{\alpha \log n} \le n^{-0.99}.$$

(b) By (a), $\mu_i(s,j)$ decreases with growing s. Thus, similarly to (a), since

$$ij^{r-1} \le ik_i^{r-1} \le \alpha \log n,$$

we have

$$\mu_i(s,j) \ge \mu_i(j-1,j) = (n-j+1)p_i^{\binom{j-1}{r-1}} \ge \frac{n}{2}\left(p/e^{i-1}\right)^{j^{r-1}} \ge \frac{n}{2}(p/e)^{\alpha \log n} \ge n^{0.99},$$

for sufficiently small α. ∎

Finally, as an important part of the forthcoming lemma is a sequence of r-graphs $G_1,\dots,G_{i_0}$, for given $0 \le s < j \le k_i$ and $S \in \binom{V}{s}$, we denote by $N_i(S,j)$ the number of (S,j)-cliques in G_i. Note that $N_1(S,s+1)$ is the deterministic counterpart of the random variable $X(S)$ appearing in Claim 2. In particular, if $G_1 \in \mathcal{A}$, then

$$|N_1(S,s+1) - \mu_1(s,s+1)| \le n^{-1/3}\mu_1(s,s+1). \tag{5}$$

Note also that, for $s=0$, $N_i(\emptyset,j)$ is just the number of cliques of order j in G_i, in particular, $N_i(\emptyset,r)=|G_i|$, the number of edges of G_i. (From now on, we will denote the number of edges of an r-graph G by $|G|$.) Finally, notice that by a comment at the end of Section 1 and by (3), for $j<r$,

$$N_i(S,j) = \binom{n-s}{j-s} = \mu_i(s,j). \tag{6}$$

3.2. Statement of Lemma 4 and proof of Theorem 1

Here we state a crucial, technical lemma from which Theorem 1 will follow. Out of the three properties listed therein, the second one, $\mathcal{Q}_i$, is there just to facilitate the proof. All parameters appearing in the statement have been defined in the previous subsection.

Lemma 4. *For every n-vertex r-graph $G_1 \in \mathcal{A}$, where the event $\mathcal{A}$ is defined in Claim 2, there exist a descending sequence of r-graphs*

$$G_1 \supset G_2 \supset \cdots \supset G_{i_0}$$

and an ascending sequence of families of cliques

$$\emptyset = \mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_{i_0}$$

such that the three properties below hold.

($\mathcal{P}_i$) *For all $2 \le i \le i_0$, $\mathcal{C}_i$ is a clique cover of $G_1 - G_i$ and*

$$|\mathcal{C}_i \setminus \mathcal{C}_{i-1}| \le \frac{2p_{i-1}n^r}{k_{i-1}^r}.$$

($\mathcal{Q}_i$) *For all $1 \le i \le i_0$, all $0 \le s \le k_i - 1$, and all $S \in \binom{V}{s}$,*

$$|N_i(S, s+1) - \mu_i(s, s+1)| \le in^{-1/3}\mu_i(s, s+1). \tag{7}$$

($\mathcal{R}_i$) *For all $1 \le i \le i_0$, all $0 \le s < j \le k_i$, and all $S \in \binom{V}{s}$,*

$$|N_i(S, j) - \mu_i(s, j)| \le n^{-1/4}\mu_i(s, j). \tag{8}$$

Note that for $j = s + 1$, property $\mathcal{R}_i$ is overwritten by $\mathcal{Q}_i$. This is because, $i \le i_0 \le \frac{r+1}{r-1} \log\log n$ and thus, for all n, the right-hand side of (7) is smaller than the right-hand side of (8). Also, by (6), for $j < r$ the left-hand side of (8) equals 0. For the same reason, whenever $s \le r - 2$, the left-hand side of (7) equals 0. This means that in these cases properties $\mathcal{Q}_i$ and $\mathcal{R}_i$ hold trivially.

We defer the proof of Lemma 4 for later. Now, we give a short proof of Theorem 1 based on Claim 2 and Lemma 4.

***Proof of Theorem* 1.** By Claim 2, the random r-graph $G_1 = G^{(r)}(n, p)$ a.a.s. satisfies event $\mathcal{A}$ and so, we are in position to fix $G_1 \in \mathcal{A}$ and apply Lemma 4. Obviously, the union $\mathcal{C}$ of the clique cover $\mathcal{C}_{i_0}$ and the edge set of the graph G_{i_0} form a clique cover of G_1. Further, recalling that $|G_{i_0}| = N_{i_0}(\emptyset, r)$, we have, by

$\mathcal{P}_i$, $i = 2, \dots, i_0$, and by $\mathcal{R}_{i_0}$ applied only in one special case of $S = \emptyset$, $j = r$, and $i = i_0$,

$$|\mathcal{C}| = |\mathcal{C}_{i_0}| + |G_{i_0}| = \sum_{i=1}^{i_0-1} |\mathcal{C}_{i+1} \setminus \mathcal{C}_i| + N_{i_0}(\emptyset, r) \le \sum_{i=1}^{i_0-1} \frac{2p_i n^r}{k_i^r} + (1 + n^{-1/4})\mu_{i_0}(0, r)$$
$$\le \sum_{i=1}^{i_0-1} \frac{2pn^r i^{\frac{r}{r-1}}}{e^{i-1}(\alpha \log n)^{r/(r-1)}} + 2\binom{n}{r} p_{i_0} \le \frac{2epn^r}{(\alpha \log n)^{r/(r-1)}} \sum_{i=1}^{\infty} \frac{i^{\frac{r}{r-1}}}{e^i} + \frac{2epn^r}{e^{i_0}}$$
$$\le \frac{2epCn^r}{(\alpha \log n)^{r/(r-1)}} + \frac{2epn^r}{(\alpha \log n)^{(r+1)/(r-1)}} = \frac{2epC(1 + o(1))n^r}{(\alpha \log n)^{r/(r-1)}},$$

where $C = \sum_{i=1}^{\infty} \frac{i^{\frac{r}{r-1}}}{e^i}$. ■

4. Preparations for the Proof of Lemma 4

First, we are going to show that property $\mathcal{R}_i$ is, in some sense, redundant. Nevertheless, we found it convenient to state it explicitly in Lemma 4.

4.1. $\mathcal{Q}_i$ implies $\mathcal{R}_i$

This short subsection is devoted to proving the following implication.

Claim 5. *For all* $1 \le i \le i_0$*, if* $G_i \in \mathcal{Q}_i$*, then* $G_i \in \mathcal{R}_i$

Proof. The key idea is to view (S, j)-cliques as a result of an iterative process of vertex by vertex "extensions" of the set S (with all required edges present). Fix $0 \le s < j \le k$, $S = \{v_1, \dots, v_s\} \in \binom{V}{s}$ and let $\mathcal{N}_i(S, j)$ be the set of all (S, j)-cliques in G_i. Recall that $N_i(S, j) = |\mathcal{N}_i(S, j)|$, that is, $N_i(S, j)$ counts the number of sets $\{v_{s+1}, \dots, v_j\} \subset V \setminus S$ such that $J = S \cup \{v_{s+1}, \dots, v_j\}$ is an (S, j)-clique in G_i. Similarly, we define $N'_i(S, j)$ as the number of *sequences* $(v_{s+1}, \dots, v_j)$ of $j - s$ *distinct* vertices in $V \setminus S$ such that, again, $J = S \cup \{v_{s+1}, \dots, v_j\}$ is an (S, j)-clique in G_i. Equivalently,

$$N'_i(S, j) = |\{(v_{s+1}, \dots, v_j) : S \cup \{v_{s+1}, \dots, v_j\} \in \mathcal{N}_i(S, j)\}|.$$

We have, obviously,

$$N'_i(S, j) := (j - s)! N(S, j). \tag{9}$$

For all $s + 1 \le \ell \le j$ and $v_{s+1}, \dots, v_\ell$, by property $\mathcal{Q}_i$ applied to the set $S_\ell := S \cup \{v_{s+1}, \dots, v_\ell\}$, setting $V_\ell = \{v_{\ell+1} : S_\ell \cup \{v_{\ell+1}\} \in \mathcal{N}_i(S_\ell, \ell + 1)\}$,

$$\Big| |V_\ell| - \mu_i(s + \ell, s + \ell + 1) \Big| \le in^{-1/3} \mu_i(s + \ell, s + \ell + 1).$$

Note that, by definition (3) and the standard combinatorial identity $\sum_{h=0}^{t-1}\binom{h}{r-1} = \binom{t}{r}$,

$$\prod_{h=s}^{j-1}\mu_i(h,h+1) = (n-s)_{(j-s)}p_i^{\sum_{h=s}^{j-1}\binom{h}{r-1}} = (j-s)!\mu_i(s,j).$$

Thus, observing that $N_i'(S,j) = \prod_{\ell=s}^{j-1}|V_\ell|$, we arrive at

$$\left(1-in^{-1/3}\right)^{j-s}(j-s)!\mu_i(s,j) \le N_i'(S,j) \le \left(1+in^{-1/3}\right)^{j-s}(j-s)!\mu_i(s,j).$$

Comparing with (9) and canceling $(j-s)!$ sidewise, this yields

$$\left(1-in^{-1/3}\right)^{j-s}\mu_i(s,j) \le N_i(S,j) \le \left(1+in^{-1/3}\right)^{j-s}\mu_i(s,j).$$

Finally, as $(j-s)in^{-1/3} \le k_i(i_0+1)n^{-1/3} = o(n^{-1/4})$, we conclude that

$$|N_i(S,j) - \mu_i(s,j)| \le n^{-1/4}\mu_i(s,j)$$

which means that G_i, indeed, satisfies property $\mathcal{R}_i$. ■

4.2. A random procedure

We intend to prove Lemma 4 by induction on i. Suppose that for some $1 \le i \le i_0 - 1$, a graph G_i and a clique cover $\mathcal{C}_i$ of $G_1 - G_i$ satisfy properties $\mathcal{P}_i$, $\mathcal{Q}_i$, and $\mathcal{R}_i$. To obtain $(G_{i+1}, \mathcal{C}_{i+1})$, we apply a random procedure during which we simultaneously select

- $\mathcal{K}_i$ — a random collection of cliques in G_i of order k_i, each chosen independently with probability

$$q_i := \frac{1}{(1+n^{-1/4})\mu_i(r,k_i)} \tag{10}$$

  and

- $\mathcal{E}_i$ — a random collection of edges $f \in G_i$, viewed as r-vertex cliques, each f chosen independently with probability

$$q_{i,f} = 1 - (1-q_i)^{(1+n^{-1/4})\mu_i(r,k_i) - N_i(f,k_i)}. \tag{11}$$

  Then, we set

- $\mathcal{C}_{i+1} := \mathcal{C}_i \cup \mathcal{K}_i \cup \mathcal{E}_i$, and
- $G_{i+1} = G_i - (\bigcup\mathcal{K}_i \cup \mathcal{E}_i)$, where $\bigcup\mathcal{K}_i$ is the set of edges covered by the union of cliques in $\mathcal{K}_i$.

(The idea of using such a random procedure has appeared in a similar context already in [1].)

The selections of $\mathcal{K}_i$ and $\mathcal{E}_i$ are performed simultaneously, that is, independently of each other. Note also that the exponent in (11) is, due to property $\mathcal{R}_i$, nonnegative. Finally, observe that for an edge $f \in G_i$, the probability that $f \in G_{i+1}$ equals

$$(1-q_i)^{N_i(f,k_i)}(1-q_{i,f}) = (1-q_i)^{(1+n^{-1/4})\mu_i(r,k_i)} \sim \frac{1}{e},$$

which explains the definition of p_i given earlier.

4.3. $\mathcal{R}_i$ implies $\mathcal{P}_{i+1}$

The following result is the first ingredient of the forthcoming probabilistic proof of Lemma 4.

Claim 6. *For all $1 \le i \le i_0 - 1$, if $G_i \in \mathcal{R}_i$, then, with probability at least 0.49, the pair $(G_{i+1}, \mathcal{C}_{i+1})$ satisfies property $\mathcal{P}_{i+1}$.*

Proof. Recall that, by the random procedure described in Subsection 4.2,

$$\mathcal{C}_{i+1} \setminus \mathcal{C}_i = \mathcal{K}_i \cup \mathcal{E}_i, \tag{12}$$

where $\mathcal{K}_i$ is a collection of k_i-cliques and $\mathcal{E}_i$ is a collection of edges selected randomly and independently from G_i.

As each k_i-clique is drawn with the same probability q_i, the quantity $|\mathcal{K}_i|$ is binomially distributed with expectation $\mathbf{E}|\mathcal{K}_i| = N_i(0,k_i) \times q_i$. This, for large n can be estimated, using property $\mathcal{R}_i$, the definition (3) of $\mu_i(s,j)$, and the divergence $k_i \to \infty$ as $n \to \infty$ (cf. definitions of k_i and i_0 in Subsection 3.1), as follows:

$$\begin{aligned}\mathbf{E}|\mathcal{K}_i| = N_i(0,k_i) \times q_i = \frac{N_i(0,k_i)}{(1+n^{-1/4})\mu_i(r,k_i)} \le \frac{\mu_i(0,k_i)}{\mu_i(r,k_i)} = \frac{(n)_r}{(k_i)_r} p_i \\ \le \left(\frac{n}{k_i - r + 1}\right)^r p_i = (1+o(1))(n/k_i)^r p_i \le 1.01 (n/k_i)^r p_i.\end{aligned}$$

Similarly, quantity $|\mathcal{E}_i|$ has a general binomial distribution with

$$\mathbf{E}|\mathcal{E}_i| = \sum_{f \in G_i} q_{i,f}. \tag{13}$$

For $f \in G_i$, by property $\mathcal{R}_i$ and Bernoulli's inequality, we have

$$q_{i,f} \le 1 - (1-q_i)^{2n^{-1/4}\mu_i(r,k_i)} \le 2n^{-1/4}\mu_i(r,k_i) q_i \le 2n^{-1/4}. \tag{14}$$

Consequently, bounding crudely $|G_i| \le n^r$, by (13) and (14),

$$\mathbf{E}|\mathcal{E}_i| \le n^r \times 2n^{-1/4} = O(n^{r-1/4}).$$

Further, observe that by the definitions of p_i, k_i, and i_0,

$$\frac{k_i^r}{p_i} \le \frac{e^i k_i^r}{pe} \le \frac{e^{i_0} k_i^r}{pe} = O\left((\log n)^{\frac{r+1}{r-1}+\frac{r}{r-1}}\right) = O\left((\log n)^{\frac{2r+1}{r-1}}\right).$$

Thus, $\mathbf{E}|\mathcal{E}_i| = O(n^{r-1/4}) = o\left((n/k_i)^r p_i\right)$, and, by (12), we have

$$\mathbf{E}|\mathcal{C}_{i+1} \setminus \mathcal{C}_i| = \mathbf{E}|\mathcal{K}_i| + \mathbf{E}|\mathcal{E}_i| \le (1.01 + o(1))(n/k_i)^r p_i.$$

Finally, by Markov's inequality,

$$\text{Prob}(|\mathcal{C}_{i+1} \setminus \mathcal{C}_i| > 2(n/k_i)^r p_i) \le 0.51.$$

It means that property $\mathcal{P}_{i+1}$ holds for $(G_{i+1}, \mathcal{C}_{i+1})$ with probability at least 0.49.

■

5. Proof of Lemma 4

We are going to show the existence of sequences $G_1 \supset G_2 \supset \cdots \supset G_{i_0}$ and $\emptyset = \mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_{i_0}$ satisfying properties $\mathcal{P}_i$, $\mathcal{Q}_i$, and $\mathcal{R}_i$, by induction on $i = 1, \dots, i_0$.

Let us begin with the base case $i = 1$. For a fixed $G_1 \in \mathcal{A}$, property $\mathcal{Q}_1$ follows from Claim 2 (cf. (5)), while property $\mathcal{R}_1$ is implied by $\mathcal{Q}_1$, as shown in Claim 5.

Assuming now that for some $i \ge 1$ a pair $(G_i, \mathcal{C}_i)$ satisfies properties $\mathcal{P}_i$ (only for $i \ge 2$), $\mathcal{Q}_i$, and $\mathcal{R}_i$. We are going to show that with positive probability the pair $(G_{i+1}, \mathcal{C}_{i+1})$, chosen randomly according to the procedure described in Subsection 4.2, satisfies $\mathcal{P}_{i+1}$, $\mathcal{Q}_{i+1}$, and $\mathcal{R}_{i+1}$, and thus, such a pair exists.

By Claim 5, property $\mathcal{Q}_{i+1}$ implies property $\mathcal{R}_{i+1}$. Moreover, by Claim 6, property $\mathcal{P}_{i+1}$ holds for $(G_{i+1}, \mathcal{C}_{i+1})$ with probability at least 0.49. Thus, it suffices to prove that G_{i+1} satisfies property $\mathcal{Q}_{i+1}$ with probability strictly greater than 0.51. In fact, the latter probability will turn out to be $1 - o(1)$.

We begin with estimating the expectation of

$$X := N_{i+1}(S, s+1)$$

the (random) number of $(S, s+1)$-cliques in G_{i+1}.

Claim 7. *For all $1 \le i \le i_0 - 1$, if $G_i \in \mathcal{Q}_i$, then, for all $r - 1 \le s < k_i$, and all $S \in \binom{V}{s}$,*

$$|\mathbf{E}X - \mu_{i+1}(s, s+1)| \le (i + 0.5)n^{-1/3}\mu_{i+1}(s, s+1).$$

Proof. Fix $r-1 \le s < k_i$ and $S \in \binom{V}{s}$ and recall our notation E_J for the set of all edges of an (S,j)-clique J and $\mathcal{N}_i(S,j)$ for the family of all (S,j)-cliques in G_i. By linearity of expectation,

$$\mathbf{E}X = \sum_{J \in \mathcal{N}_i(S,s+1)} \mathrm{Prob}(E_J \subset G_{i+1}). \tag{15}$$

To estimate $\mathrm{Prob}(E_J \subset G_{i+1})$, observe that an $(S,s+1)$-clique J of G_i "survives" into G_{i+1} if none of its edges was selected to $\mathcal{E}_i$ or belonged to some k_i-clique selected to $\mathcal{K}_i$. The probability of the former event is $\prod_{f\in E_J}(1-q_{i,f})$, while the probability of the latter event is $(1-q_i)^{|\mathcal{U}|}$, where $\mathcal{U} := \bigcup_{f\in E_J} \mathcal{N}_i(f,k_i)$. Set

$$m_1 = |\mathcal{U}| - \sum_{f\in E_J} N_i(f,k_i) \quad \text{and} \quad m_2 = (1+n^{-1/4})\mu_i(r,k_i)|E_J|.$$

Then, using (11), we infer that

$$\mathrm{Prob}(E_J \subset G_{i+1}) = (1-q_i)^{|\mathcal{U}|} \prod_{f\in E_J}(1-q_{i,f}) = (1-q_i)^{m_1+m_2}. \tag{16}$$

Next, we separately find lower and upper bounds on $(1-q_i)^{m_1}$ and $(1-q_i)^{m_2}$. By Bonferroni's inequality, property $\mathcal{R}_i$ (which follows from $\mathcal{Q}_i$, see Claim 5), and the monotonicity of $\mu_i(t,k_i)$ as a function of t (see Claim 3(a)), the quantity $-m_1$ can be bounded as follows:

$$\begin{aligned} 0 \le -m_1 &\le \sum_{g,h\in E_J, g\neq h} N_i(g\cup h, k_i) \le \sum_{g,h\in E_J, g\neq h} (1+n^{-1/4})\mu_i(|g\cup h|, k_i) \\ &\le \sum_{g,h\in E_J, g\neq h} (1+n^{-1/4})\mu_i(r+1,k_i) \le |E_J|^2(1+n^{-1/4})\mu_i(r+1,k_i). \end{aligned} \tag{17}$$

(Above, we maximized $\mu_i(|g\cup h|,k_i)$ by minimizing $|g\cup h|$ which achieves minimum at $r+1$.) Note that

$$|E_J| = \binom{s}{r-1} \le k^{r-1} = \Theta(\log n). \tag{18}$$

Consequently, by Claim 3(a) and the definition (10) of q_i,

$$\begin{aligned} 1 &\le (1-q_i)^{m_1} \le \exp\left\{q_i|E_J|^2\left(1+n^{-1/4}\right)\mu_i(r+1,k_i)\right\} \\ &= \exp\left\{\frac{|E_J|^2\mu_i(r+1,k_i)}{\mu_i(r,k_i)}\right\} \overset{Cl.3(\mathrm{a})}{\le} \exp\left\{|E_J|^2 n^{-0.99}\right\} = 1 + o\left(n^{-0.98}\right). \end{aligned} \tag{19}$$

Further, by Claim 3(b), (10), and (18),

$$\tfrac{q_i|E_J|}{1-q_i} \le 2q_i|E_J| = \frac{O(\log n)}{\left(1+n^{-1/4}\right)\mu_i(r,k_i)} \le 2n^{-0.99}\Theta(\log n) = o\left(n^{-0.98}\right).$$

This implies that

$$\begin{aligned} e^{-|E_J|} &\ge (1-q_i)^{m_2} \ge \exp\left\{-\frac{|E_J|}{1-q_i}\right\} \ge e^{-|E_J|}\left(1-\frac{q_i|E_J|}{1-q_i}\right) \\ &\ge e^{-|E_J|}\left(1-o(n^{-0.98})\right). \end{aligned} \tag{20}$$

Thus, by (16), (19), and (20),

$$\left(1-o(n^{0.98})\right)e^{-|E_J|} \le \mathrm{Prob}(E_J \subset G_{i+1}) \le \left(1+o(n^{-0.98})\right)e^{-|E_J|}. \tag{21}$$

Recall that, by property $\mathcal{Q}_i$, $N_i(S,s+1) \le (1+in^{-1/3})\mu_i(s,s+1)$, while, by (4), $\mu_i(s,s+1)e^{-\binom{s}{r-1}} = \mu_{i+1}(s,s+1)$. Thus, using also (15) and (21), and recalling that $|E_J| = \binom{s}{r-1}$, we finally have

$$\begin{aligned} \mathbf{E}X &\le N_i(S,s+1)\left(1+o(n^{-0.98})\right)e^{-\binom{s}{r-1}} \\ &\le \left(1+o(n^{-0.98})\right)(1+in^{-1/3})\mu_i(s,s+1)e^{-\binom{s}{r-1}} \\ &\overset{(4)}{=} \left(1+o(n^{-0.98})\right)\left(1+in^{-1/3}\right)\mu_{i+1}(s,s+1) \\ &\le \left(1+(i+0.5)n^{-1/3}\right)\mu_{i+1}(s,s+1) \end{aligned}$$

and, similarly, $\mathbf{E}X \ge \left((1-(i+0.5)n^{-1/3}\right)\mu_{i+1}(s,s+1)$. ■

In view of the above claim, to establish property $\mathcal{Q}_{i+1}$ of G_{i+1}, it remains to show that X is concentrated around its expectation with probability very close to 1. In doing so, similarly to [4], we will utilize the following Azuma-type concentration inequality which can be deduced from [7], Theorem 3.8 (see also [8], Corollary 1.4).

Lemma 8. *Let $X_1,\dots,X_M$ be 0–1 independent random variables and let $f : \{0,1\}^{[M]} \to \mathbb{R}$ satisfy Lipschitz condition (L) with constants $c_1,\dots,c_M$:*

(L) for all $(z_1,\dots,z_M) \in \{0,1\}^{[M]}$ and $(z'_1,\dots,z'_M) \in \{0,1\}^{[M]}$, and all $1 \le m \le M$,

$$|f(z_1,\dots,z_M) - f(z'_1,\dots,z'_M)| \le c_m, \quad \textit{whenever } z_h = z'_h \textit{ for all } h \ne m.$$

Set

$$X = f(X_1,\dots,X_M), \quad W = \sum_{m=1}^{M} c_m^2\,\mathrm{Prob}(X_m=1), \quad \textit{and} \quad C = \max_{1\le m\le M} c_m.$$

Then, for every $t \geq 0$,

$$\mathrm{Prob}(|X - \mathbf{E}X| \geq t) \leq 2\exp\left\{-\frac{t^2}{2(W+Ct)}\right\}.$$ ■

Now, we are ready to provide the last ingredient of the proof of Lemma 4.

Claim 9. *For all $1 \leq i \leq i_0 - 1$, if $G_i \in \mathcal{Q}_i$, then, a.a.s., $G_{i+1} \in \mathcal{Q}_{i+1}$.*

Proof. Fix $r-1 \leq s < k_i$ and $S \in \binom{V}{s}$, and notice that if J is an $(S, s+1)$-clique in G_{i+1}, then it must have also been an $(S, s+1)$-clique in G_i, whose all edges "survived" the random procedure described in Subsection 4.2.

Recall that $\mathcal{N}_i(\emptyset, k_i)$ denotes the set of all k_i-cliques in G_i and that $N_i(\emptyset, k_i) = |\mathcal{N}_i(\emptyset, k_i)|$. We set $M_1 = N_i(\emptyset, k_i)$ and $\mathcal{N}_i(\emptyset, k_i) = \{K_1, \ldots, K_{M_1}\}$. Let X_m, $m = 1, \ldots, M_1$, be the indicator random variable which equals 1 if $K_m \in \mathcal{K}_i$ and 0 otherwise. Similarly, set $M_2 = |G_i|$ and $G_i = \{f_1, \ldots, f_{M_2}\}$, and denote by Y_m, $m = 1, \ldots, M_2$, the indicator random variable equal to 1 if $f_m \in \mathcal{E}_i$ and 0 otherwise.

As the events $K_m \in \mathcal{K}_i$, $m = 1, \ldots, M_1$, and $f_m \in \mathcal{E}_i$, $m = 1, \ldots, M_2$, fully determine the number of (S, s)-cliques left in G_{i+1}, there exists a function $f : \{0,1\}^{[M_1+M_2]} \to \mathbb{R}$, such that

$$X = N_{i+1}(S, s+1) = f(X_1, \ldots, X_{M_1}, Y_1, \ldots, Y_{M_2}).$$

The explicit form of function f is not important for us.

As we are aiming at applying Lemma 8 to X, we need to find constants for which the Lipschitz condition (L) holds and then estimate W. Set

$$c_m = \max |f(x_1, \ldots, x_{M_1}, y_1, \ldots, y_{M_2}) - f(x'_1, \ldots, \ldots, x'_{M_1}, y_1, \ldots, y_{M_2})|,$$

where the maximum is taken over all $(x_1, \ldots, x_{M_1})$, $(x'_1, \ldots, x'_{M_1}) \in \{0,1\}^{[M_1]}$ and $(y_1, \ldots, y_{M_2}) \in \{0,1\}^{[M_2]}$ such that $x_h = x'_h$ for all $h \neq m$. Similarly, we set

$$d_m = \max |f(x_1, \ldots, x_{M_1}, y_1, \ldots, y_{M_2}) - f(x_1, \ldots, \ldots, x_{M_1}, y'_1, \ldots, y'_{M_2})|,$$

where the maximum is taken over all $(x_1, \ldots, x_{M_1}) \in \{0,1\}^{[M_1]}$ and $(y_1, \ldots, y_{M_2})$, $(y'_1, \ldots, y'_{M_2}) \in \{0,1\}^{[M_2]}$ such that $y_h = y'_h$ for all $h \neq m$. In other words, c_m and d_m are, respectively, upper bounds on the change of X due to flipping the outcome of the event $K_m \in \mathcal{K}_i$, respectively, $f_m \in \mathcal{E}_i$.

Now we estimate the Lipschitz parameters c_m and d_m taking into account the position of K_m and f_m with respect to the given set S. We begin with d_m as this case is easier. As an edge of G_i may belong to at most one $(S, s+1)$-clique

J, the values of $X = N_{i+1}(S, s+1)$ for G_{i+1} with or without the edge f_m may differ by at most one. Thus,

$$d_m \leq 1 \tag{22}$$

for all $m = 1, \dots, M_2$.

On the other hand, since $|J| = s+1$, any such J contains exactly $\binom{s}{r-1}$ edges of G_i, so there are altogether

$$N_i(S, s+1) \times \binom{s}{r-1} \leq N_i(S, s+1)k^{r-1} \tag{23}$$

edges $f_m \in G_i$ whose removal could affect $X = N_{i+1}(S, s+1)$. Thus, for that many edges we put $d_m = 1$, while for all other edges $d_m = 0$.

Turning to c_m, by the same token, very crudely, for all $m = 1, \dots, M_1$,

$$c_m \leq |K_m| = \binom{k_i}{r} \leq k^r, \tag{24}$$

as every edge of K_m may belong to at most one $(S, s+1)$-clique J.

Moreover, $c_m > 0$ only if K_m contains at least one edge of some $(S, s+1)$-clique of G_i. There are $N_i(S, s+1)$ such $(S, s+1)$-cliques and each contains $\binom{s}{r-1}$ edges. In turn, by property $\mathcal{R}_i$, each edge f is contained in $N_i(f, k_i) \leq (1 + n^{-1/4})\mu_i(r, k_i)$ k_i-cliques of G_i. Hence, there are at most

$$N_i(S, s+1) \times \binom{s}{r-1} \times \max_{f \in G_i} N_i(f, k_i) \leq N_i(S, s+1)k^{r-1}(1+n^{-1/4})\mu_i(r, k_i) \tag{25}$$

cliques K_m in G_i which share an edge with some $(S, s+1)$-clique. This implies that for at most that many indices $m \in [M_1]$ we have $c_m > 0$.

Putting (22)–(25) together, one can bound the parameter W appearing in Lemma 8, using again property $\mathcal{R}_i$, the definition (10) of q_i, and the estimate (14) of $q_{i,f}$, as follows.

(26)
$$W = \sum_{m=1}^{M_1} c_m^2 q_i + \sum_{m=1}^{M_2} d_m^2 q_{i,f} \overset{(14)}{\leq} N_i(S, s+1)k^{r-1}(1 + n^{-1/4})\mu_i(r, k_i) \times k^{2r} \times q_i$$
$$+ N_i(S, s+1)k^{r-1} \times 1^2 \times 2n^{-1/4} \overset{(10)}{\leq} (1+o(1))N_i(S, s+1)k^{3r-1} \overset{\mathcal{R}_i}{\leq} \mu_i(s, s+1)k^{3r}.$$

Recall that, by definition (3), $\mu_i(s, s+1) = (n-s)p_i^{\binom{s}{r-1}} \leq n$, while, by equality (4), $\mu_{i+1}(s, s+1) = e^{-\binom{s}{r-1}}\mu_i(s, s+1) \leq \mu_i(s, s+1)$. Moreover, by Claim 3(b) applied with $i+1$, $\mu_{i+1}(s, s+1) \geq n^{0.99}$. Putting all these facts together, we have

$$n^{0.99} \leq \mu_{i+1}(s, s+1) \leq \mu_i(s, s+1) \leq n. \tag{27}$$

Thus, in view of (26), by Lemma 8 with

$$t = \frac{1}{2}n^{-1/3}\mu_{i+1}(s, s+1) \quad \text{and} \quad C = \max\left\{\max_{1\le m\le M_1} c_m, \max_{1\le m\le M_2} d_m\right\} \le k^r,$$

noting that $Ct = o\left(\mu_i(s, s+1)k^{3r}\right)$ and taking n sufficiently large,

$$\text{Prob}(X - \mathbf{E}X \ge t) \le 2\exp\left\{-\frac{\frac{1}{4}n^{-2/3}\mu_{i+1}^2(s, s+1)}{2(\mu_i(s, s+1)k^{3r} + Ct)}\right\}$$
$$\le 2\exp\left\{-\frac{\mu_{i+1}^2(s, s+1)}{9\mu_i(s, s+1)n^{2/3}k^{3r}}\right\} \overset{(27)}{\le} 2\exp\left\{-\frac{n^{1.98}}{n^{5/3}k^{3r}}\right\} \le \exp\left\{-n^{0.3}\right\}.$$

In view of the above and using Claim 7 and the union bound, a.a.s., for all s and $S \in \binom{V}{s}$,

$$X = N_{i+1}(S, s+1) \le \mathbf{E}X + t \le \left(1 + (i+0.5)n^{-1/3} + 0.5n^{-1/3}\right)\mu_{i+1}(s, s+1)$$
$$\le \left(1 + (i+1)n^{-1/3}\right)\mu_{i+1}(s, s+1)$$

and, similarly, $X \ge \left(1 - (i+1)n^{-1/3}\right)\mu_{i+1}(s, s+1)$, which completes the proof of Claim 9. ■

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Appendix

The following folklore result was observed by many authors for graphs ($r = 2$) but there seems to be no published proof of the general case. Here we fill that gap.

Fact 1. *Let H be an r-graph and let $\theta_1(H)$ and $\tilde{\theta}_1(H)$ stand, respectively, for its representation number and minimum (edge) clique cover number. Then $\theta_1(H) = \tilde{\theta}_1(H)$.*

For the proof we need a simple observation.

Observation. *Let $H = (V, E)$ be an r-graph and let $\mathcal{S} = \{S_v \subset S : v \in V\}$ be a representation of H with the smallest set S. Then every element $s \in S$ belongs to at least r sets in $\mathcal{S}$.*

Proof. Suppose there is an $s \in S$ belonging to fewer than r sets in $\mathcal{S}$. Then $\mathcal{S}_s = \{S_v \setminus \{s\} : v \in V\}$ would also be a representation of H which contradicts the minimality of S. Indeed, for such an s and any R with $|R| = r$,

$$\bigcap_{v \in R} S_v \neq \emptyset \quad \text{if and only if} \quad \bigcap_{v \in R} (S_v \setminus \{s\}) \neq \emptyset.$$

□

***Proof of Fact* 1.** Let $\mathcal{S} = \{S_v \subset S : v \in V\}$ be a minimum representation of H, that is, a representation of size $|\mathcal{S}| = \theta_1(H)$. By the above Observation, for each $s \in S$ the set $C(s) = \{v : s \in S_v\}$ has size $|C(s)| \geq r$. What is more important, $C(s)$ is a clique in H. Indeed, if $\{v_1, \ldots, v_r\} \subset C(s)$, then $S_{v_1} \cap \cdots \cap S_{v_r} \ni s$, thus $\{v_1, \ldots, v_r\} \in H$. Moreover, each edge $\{v_1, \ldots, v_r\} \in H$ is covered by a clique $C(s)$, where $s \in S_{v_1} \cap \cdots \cap S_{v_r}$. Hence, $\tilde{\theta}_1(H) \leq \theta_1(H)$.

Conversely, let $\{C(s) : s \in S\}$ be a clique cover of H indexed by some (abstract) set S. For every vertex $v \in V$ consider the set

$$S_v = \{s \in S : v \in C(s)\}.$$

Next, observe that $\{v_1, \dots, v_r\} \in E$ if and only if there is some $s \in S$ with $\{v_1, \dots, v_r\} \subset C(s)$. We will draw two consequences of this equivalence. First, if $\{v_1, \dots, v_r\} \in E$, then there exists $s \in S_{v_1} \cap \dots \cap S_{v_r}$, implying that

$$S_{v_1} \cap \dots \cap S_{v_r} \neq \emptyset.$$

However, if $\{v_1, \dots, v_r\} \notin E$ then $\{v_1, \dots, v_r\} \not\subset C(s)$ for all $s \in S$, which means that $S_{v_1} \cap \dots \cap S_{v_r} = \emptyset$. Consequently, $\{S_v : v \in V\}$ is a representation of H, yielding $\theta_1(H) \leq \tilde{\theta}_1(H)$. $\square$