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A SHORT PROOF OF ERDŐS' CONJECTURE FOR TRIPLE SYSTEMS

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Abstract. Erdős [1] conjectured that for all $k \geq 2$, $s \geq 1$ and $n \geq k(s+1)$, an n-vertex k-uniform hypergraph $\mathcal{F}$ with $\nu(\mathcal{F}) = s$ cannot have more than $\max\{\binom{sk+k-1}{k}, \binom{n}{k} - \binom{n-s}{k}\}$ edges. It took almost fifty years to prove it for triple systems. In [5] we proved the conjecture for all s and all $n \geq 4(s+1)$. Then Łuczak and Mieczkowska [6] proved the conjecture for sufficiently large s and all n. Soon after, Frankl proved it for all s. Here we present a simpler version of that proof which yields Erdős' conjecture for $s \geq 33$. Our motivation is to lay down foundations for a possible proof in the much harder case $k = 4$, at least for large s.

1. Introduction

Let $\nu(\mathcal{F})$ denote the size of a largest matching in a k-uniform hypergraph $\mathcal{F}$. Erdős [1] conjectured that for all $k \geq 2$, $s \geq 1$ and $n \geq k(s+1)$, an n-vertex k-uniform graph $\mathcal{F}$ with $\nu(\mathcal{F}) = s$ cannot have more than

$$\max\left\{\binom{sk+k-1}{k}, \binom{n}{k} - \binom{n-s}{k}\right\}$$

edges and proved it for $n > n_0(k, s)$. So far, the best general upper bound, $n_0(k, s) \leq (2s+1)k - s$, is due to Frankl [3]. While the conjecture was proved

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for $k = 2$ (i.e., for graphs) by Erdős and Gallai already in [2], progress for $k \geq 3$ has been much slower.

It took almost fifty years to prove it for triple systems ($k = 3$). First, in [5] we proved the conjecture for all s and all $n \geq 4(s + 1)$. Then Łuczak and Mieczkowska [6] proved the conjecture for sufficiently large s and all n. Soon after, Frankl [4] proved it for all s, building upon some ideas from [6]. Here we present a streamlined version of the proof in [4] which is shorter and simpler and yields the conjecture for $s \geq 33$.

Our motivation for writing this paper is also rooted in the belief that a proof along the same lines could eventually work in the case $k = 4$, at least for large s. At the time of the writing of [4] the first author had thought that the case $k = 4$ would follow from the arguments used there for $k = 3$. However, it turned out that the situation is not that simple. This means that the case $k = 4$ should be still considered *open*. Nevertheless, we do hope that the arguments of the present paper will prove helpful in the case $k = 4$ and, maybe, beyond. (At one place in the proof we rely on the above mentioned result from [5], which for $k \geq 4$ could be replaced by the general result from [3], also mentioned above.)

We will call a 3-uniform hypergraph a *3-graph*. Let

$$m(n, s) = \max \left\{ |\mathcal{F}| : |V(\mathcal{F})| = n,\ \nu(\mathcal{F}) = s \right\}.$$

Further, let

$$\mathcal{A} := \mathcal{A}(n, s) = K^3_{3s+2} \cup (n - 3s + 2)K_1$$

be the complete 3-graph on $3s + 2$ vertices, augmented by $n - 3s + 2$ isolated vertices, and let

$$\mathcal{B} := \mathcal{B}(n, s) = K^3_n - K^3_{n-s},$$

that is, $\mathcal{B}$ is the complete 3-graph on n vertices from which a complete 3-graph on $n - s$ vertices has been removed. Finally, let

$$a(s) = |\mathcal{A}(n, s)| = \binom{3s+2}{3} \quad \text{and} \quad b(n, s) = |\mathcal{B}(n, s)| = \binom{n}{3} - \binom{n-s}{3}$$

and

$$M(n, s) = \max\{a(s), b(n, s)\}.$$

Clearly, $m(n, s) \geq M(n, s)$, and for $k = 3$ the Erdős Conjecture states that $m(n, s) = M(n, s)$ for all s and all n. Note that the conjecture is trivially valid for $n \leq 3s + 2$, since then $m(n, s) \leq a(s)$. Therefore, it suffices to consider only the case $n \geq 3s + 3$. Here we prove the following result.

Theorem 1. *For all $s \geq 33$ and $n \geq 3s + 3$, $m(n, s) = M(n, s)$.*

2. Preparations for the proof

Given a linear order $\le$ on the vertex set $V(\mathcal{F})$, we define a partial order on the triples of vertices as follows. For two sets $A, B \in \binom{[n]}{3}$ we write $A \prec B$ if $A = \{a_1 < a_2 < a_3\}$, $B = \{b_1 < b_2 < b_3\}$, and $a_i \le b_i$ for all $i = 1, 2, 3$. We say that $\mathcal{F}$ is *stable (or shifted)* if whenever $A \prec B$ and $B \in \mathcal{F}$, then $A \in \mathcal{F}$.

If a 3-graph $\mathcal{F}$ is not stable, then there exists an edge B of $\mathcal{F}$ and a non-edge A with $A \prec B$. Fix $1 \le i < j \le n$ such that $i \in A \setminus B$ and $j \in B \setminus A$. Now, *simultaneously* replace every edge F containing j, but not containing i, by $(F \setminus \{j\}) \cup \{i\}$, provided $(F \setminus \{j\}) \cup \{i\} \notin \mathcal{F}$ yet. We call such an operation *an ij-shift* and denote the resulting 3-graph by $\mathrm{sh}_{ij}(\mathcal{F})$. Note that $|\mathrm{sh}_{ij}(\mathcal{F})| = |\mathcal{F}|$. It is an easy exercise (see, e.g., [6, Lemma 3]) to show that for all $1 \le i < j \le n$ we have $\nu(\mathrm{sh}_{ij}(\mathcal{F})) \le \nu(\mathcal{F})$.

Let $\mathrm{sh}(\mathcal{F})$ be a stable 3-graph obtained from $\mathcal{F}$ by a series of ij-shifts (with varying pairs ij). Then, too, $|\mathrm{sh}(\mathcal{F})| = |\mathcal{F}|$ and $\nu(\mathrm{sh}(\mathcal{F})) \le \nu(\mathcal{F})$. This means that it is sufficient to prove the Erdős Conjecture for stable 3-graphs only. For further reference, note also that if

$$\nu(\mathcal{F}) = s \text{ and } |\mathcal{F}| = m(n, s), \quad \text{then} \quad \nu(\mathrm{sh}(\mathcal{F})) = \nu(\mathcal{F}).$$

When comparing the two quantities defining $M(n, s)$, it is apparent that for smaller n we have $a(s) > b(n, s)$, while for larger n the opposite holds. Indeed, $b(n, s)$ is an increasing function of n, while $a(s)$ is a constant. An important role in our proof is played by the critical value of n at which this transition takes place. Therefore, for every s, we define

$$n_1(s) = \min\{n : a(s) \le b(n, s)\}.$$

It follows that $M(n, s) = a(s)$ for $n \le n_1(s) - 1$, while $M(n, s) = b(n, s)$ for $n \ge n_1(s)$. The importance of the parameter $n_1(s)$ is supported by the fact, observed already in [4, Fact 5.3], that if the Erdős conjecture fails for some s and n, then it must fail for that s and $n \in \{n_1(s) - 1, n_1(s)\}$. We provide the proof for completeness. Given a 3-graph $\mathcal{F}$ and a vertex $v \in V(\mathcal{F})$, let

$$\mathcal{F}(\bar{v}) = \{F \in \mathcal{F} : \ v \notin F\} \quad \text{and} \quad \mathcal{F}(v) = \big\{ F \setminus \{v\} : v \in F \in \mathcal{F} \big\}.$$

Note that $\mathcal{F}(\bar{v})$ is a 3-graph, while $\mathcal{F}(v)$ is a graph, both on the same vertex set $V(\mathcal{F}) \setminus \{v\}$.

FACT 1. *If for some s,*

$$m(n_1(s) - 1, s) = M(n_1(s) - 1, s) \quad \textit{and} \quad m(n_1(s), s) = M(n_1(s), s),$$

then $m(n, s) = M(n, s)$ for all $n \ge 3s + 3$.

Proof. Assume that

$$m(n_1(s)-1,s) = M(n_1(s)-1,s) = a(s)$$

and

$$m(n_1(s),s) = M(n_1(s),s) = b(n_1(s),s).$$

Since $a(s)$ is independent of n, it follows that $m(n,s) = a(s)$ for all $n \le n_1(s)-1$. For $n \ge n_1(s)$ we use induction on n. Assume $m(n-1,s) = b(n-1,s)$, $n \ge n_1(s)+1$, and let $\mathcal{F}$ be a stable 3-graph on vertex set $[n] = \{1,\dots,n\}$ (ordered by natural ordering) and with $\nu(\mathcal{F}) = s$. Consider the 3-graph $\mathcal{F}(\bar{n})$ and the graph $\mathcal{F}(n)$. Clearly, $|\mathcal{F}| = |\mathcal{F}(\bar{n})| + |\mathcal{F}(n)|$ and $\nu(\mathcal{F}(\bar{n})) \le \nu(\mathcal{F}) = s$, so, by assumption,

$$|\mathcal{F}(\bar{n})| \le m(n-1,s) = b(n-1,s) = \binom{n-1}{3} - \binom{n-1-s}{3}.$$

By stability of $\mathcal{F}$, we also have $\nu(\mathcal{F}(n)) \le s$. Indeed, if there was a matching $e_1,\dots,e_{s+1}$ in $\mathcal{F}(n)$, then, since $n-1 \ge 3s+3$, the triples $e_i \cup \{v_i\}$, $i = 1,\dots,s+1$, where $\{v_1,\dots,v_{s+1}\} \subset [n-1] \setminus \bigcup_{i=1}^{s+1} e_i$, would form a matching of size $s+1$ in $\mathcal{F}$, a contradiction. Thus, by the result of Erdős and Gallai from [2] quoted earlier,

$$|\mathcal{F}(n)| \le \max\left\{\binom{2s+1}{2}, \binom{n-1}{2} - \binom{n-1-s}{2}\right\} \tag{1}$$
$$= \binom{n-1}{2} - \binom{n-1-s}{2}$$

for $n \ge 3s$. Altogether,

$$|\mathcal{F}| \le \binom{n-1}{3} - \binom{n-1-s}{3} + \binom{n-1}{2} - \binom{n-1-s}{2}$$
$$= \binom{n}{3} - \binom{n-s}{3} = b(n,s). \quad \square$$

The value of $n_1(s)$ was asymptotically determined in [6] and [5]. Here we need estimates valid also for small s.

Fact 2. *For all s and n,*

(i) $n_1(s) \le 3.5s+3$, (ii) $n_1(s) \ge 3.4s+1$, (iii) $n_1(s) - n_1(s-1) \ge 2$.

Proof. All parts follow from an exact formula for $n_1(s)$ which we derive first. We look for the smallest integral solution (in n) to the inequality

$$\binom{3s+2}{3} \le \binom{n}{3} - \binom{n-s}{3} = \binom{s}{3} + \binom{s}{2}(n-s) + s\binom{n-s}{2}$$

which, after substituting $m = n - s$, becomes

$$3m^2 + 3(s-2)m - (26s^2 + 30s + 4) \geq 0.$$

Solving this quadratic inequality and setting $g(s) = 321s^2 + 324s + 84$, we derive that

$$n_1(s) = 1 + \left\lceil \frac{1}{2}s + \frac{1}{6}\sqrt{g(s)} \right\rceil. \tag{2}$$

To show (i), observe that, in view of (2), it is now equivalent to

$$n_1(s) - 1 = \left\lceil \frac{1}{2}s + \frac{1}{6}\sqrt{g(s)} \right\rceil \leq \lfloor 3.5s + 2 \rfloor,$$

which, in turn, is equivalent to

$$\frac{1}{2}s + \frac{1}{6}\sqrt{g(s)} \leq \lfloor 3.5s + 2 \rfloor.$$

Since $3.5s + 3/2 \leq \lfloor 3.5s + 2 \rfloor$, we thus get a stronger inequality

$$g(s) \leq 36(3s + 3/2)^2 = 324s^2 + 324s + 81$$

which is true for all $s \geq 1$.

Part (ii) is even easier and we leave it to the reader. For part (iii), rewriting $n_1(s-1) \leq n_1(s) - 2$, substituting formula (2), and dropping the ceilings on both sides, we obtain a stronger inequality

$$1 + \frac{1}{2}(s-1) + \frac{1}{6}\sqrt{g(s-1)} \leq 1 + \frac{1}{2}s + \frac{1}{6}\sqrt{g(s)} - 2,$$

equivalent to

$$\frac{642s + 3}{6(\sqrt{g(s-1)} + \sqrt{g(s)})} \geq \frac{3}{2}.$$

By bounding the left-hand side from below by

$$\frac{642s + 3}{12\sqrt{g(s)}}$$

we finally obtain a yet stronger inequality $(214s + 1)^2 \geq 36g(s)$, valid for all $s \geq 1$. $\square$

We say that a stable 3-graph has property ONE if $\nu(\mathcal{F}(\bar{1})) = \nu(\mathcal{F})$, that is, if there is a largest matching not covering the smallest vertex. Let

$$m_{\mathrm{ONE}}(n, s) = \max\left\{ |\mathcal{F}| : |V(\mathcal{F})| = n, \nu(\mathcal{F}) = s, \text{ and } \mathcal{F} \text{ has ONE} \right\}.$$

Clearly, $m_{\mathrm{ONE}}(n,s) \le m(n,s)$. Note, however, that while $\mathcal{A}$ has ONE, $\mathcal{B}$ does not. This means that the inequality $m_{\mathrm{ONE}}(n,s) \ge M(n,s)$ might not be true in general. On the other hand, we are going to prove that the reverse inequality is true.

LEMMA 1. *$m_{\mathrm{ONE}}(n,s) \le M(n,s)$ for all $s \ge 25$ and all $n \ge 3s+3$.*

Lemma 1 is the main ingredient of our proof of Theorem 1. The other ingredient is the following lemma.

LEMMA 2. *If, for some s_0, $m_{\mathrm{ONE}}(n,s) \le M(n,s)$ for all $s \ge s_0$ and all $n \ge 3s+3$, then $m(n,s) = M(n,s)$ for all*

$$s \ge \tfrac{5}{4}(s_0+1) \tag{3}$$

and all $n \ge 3s+3$.

PROOF OF THEOREM 1. By Lemma 1, the assumption of Lemma 2 is satisfied with $s_0 = 25$. Then Theorem 1 follows from Lemma 2 with $s_0 = 25$, as the right-hand side of (3) equals $\frac{65}{2}$. $\square$

The proof of Lemma 2 is given below, while the proof of Lemma 1 is deferred to Section 3.

The proof of Lemma 2 is based on a fact similar to Fact 1.

FACT 3. *For every s, if $m_{\mathrm{ONE}}(n_1(s)-1,s) \le a(s)$ and $m_{\mathrm{ONE}}(n_1(s),s) \le b(n_1(s),s)$, then $m_{\mathrm{ONE}}(n,s) \le M(n,s)$ for all $n \ge 3s+3$.*

PROOF. For $n \le n_1(s)-2$, consider a stable 3-graph $\mathcal{F}$ on n vertices, with property ONE, with $|\mathcal{F}| = m_{\mathrm{ONE}}(n,s)$, and with $\nu(\mathcal{F}) = s$. By adding to it vertices $n+1, \ldots, n_1(s)-1$, we obtain a 3-graph $\mathcal{F}'$ which is still stable, has property ONE and $\nu(\mathcal{F}) = s$. Thus

$$m_{\mathrm{ONE}}(n,s) = |\mathcal{F}| = |\mathcal{F}'| \le m_{\mathrm{ONE}}(n_1(s)-1,s) = a(s) = M(n,s).$$

For $n \ge n_1(s)+1$, we apply induction on n. Assume that $m_{\mathrm{ONE}}(n-1,s) \le M(n-1,s)$ and let $\mathcal{F}$, with $|\mathcal{F}| = m_{\mathrm{ONE}}(n,s)$, be a stable 3-graph which has property ONE. To show that $m_{\mathrm{ONE}}(n,s) \le M(n,s)$, we proceed as in the proof of Fact 1. The only novelty is to observe that if $\mathcal{F}$ has property ONE, then the same is true for $\mathcal{F}(\bar{n})$. This follows, since, due to stability of $\mathcal{F}$ and the fact that $n \ge 3s+3$, there is a matching of size s in $\mathcal{F}$, avoiding vertices 1 and n. Hence, $|\mathcal{F}(\bar{n})| \le m_{\mathrm{ONE}}(n-1,s) \le M(n-1,s) \le b(n-1,s)$. As we also have (1), the conclusion follows. $\square$

PROOF OF LEMMA 2. Suppose that for some $s \ge \frac{5}{4}(s_0+1)$ and $n \ge 3s+3$, we have $m(n,s) > M(n,s)$. By Fact 1, it means that there is $n \in \{n_1(s)-1, n_1(s)\}$ and a 3-graph $\mathcal{F}$ with $V(\mathcal{F}) = [n]$, $\nu(\mathcal{F}) = s$, and $|\mathcal{F}| = m(n,s) > M(n,s)$. Let $\mathcal{F}' = \mathrm{sh}(\mathcal{F})$. Recall that $|\mathcal{F}'| = |\mathcal{F}|$ and $\nu(\mathcal{F}') = \nu(\mathcal{F}) = s$.

Since $s \geq s_0$, $\mathcal{F}'$ cannot have property ONE, since otherwise we would arrive at a contradiction with the assumption of Lemma 2.

For $q \geq 1$, denote by $\mathcal{F}'_q$ the induced sub-3-graph of $\mathcal{F}'$ obtained by removing the vertices $1, 2, \ldots, q$ and all edges adjacent to them. Observe that $\mathcal{F}'_q$ is stable and $\nu(\mathcal{F}'_q) \geq s - q$, and, by previous paragraph, $\nu(\mathcal{F}'_1) = s - 1$. Observe also that $\nu(\mathcal{F}'_s) = 0$ would mean $\mathcal{F}'_s = \emptyset$, and consequently $|\mathcal{F}| = |\mathcal{F}'| \leq b(n, s)$, which would be a contradiction with our choice of $\mathcal{F}$. Hence, $\nu(\mathcal{F}_s) \geq 1$ and, so, for some $1 \leq q \leq s - 1$, we must have $\nu(\mathcal{F}'_q) = \nu(\mathcal{F}'_{q+1})$, meaning that $\mathcal{F}'_q$ has property ONE. Let

$$q_0 = \min\{q : \mathcal{F}'_q \text{ has ONE}\}.$$

We have $1 \leq q_0 \leq s - 1$, but, in fact, q_0 is much smaller.

FACT 4. $q_0 \leq 0.2s + 1$

PROOF. We first claim that

$$n - q_0 \leq 4(s - q_0 + 1) - 1. \tag{4}$$

Indeed, we have

$$n' := |V(\mathcal{F}'_{q_0})| = n - q_0 \quad \text{and} \quad s' := \nu(\mathcal{F}'_{q_0}) = s - q_0.$$

If (4) would not hold then $n' \geq 4(s' + 1)$, and by the result from [5] mentioned in the Introduction, we would have $|\mathcal{F}'_{q_0}| \leq b(n - q_0, s - q_0) = \binom{n-q_0}{3} - \binom{n-s}{3}$ and thus,

$$|\mathcal{F}'| \leq \binom{n}{3} - \binom{n - q_0}{3} + |\mathcal{F}'_{q_0}| \leq \binom{n}{3} - \binom{n - s}{3} = M(n, s),$$

a contradiction. Consequently, (4) holds, that is, $n - q_0 \leq 4(s - q_0) + 3$, and since by Fact 2(ii), $n \geq n_0(s) - 1 \geq 3.4s$, we have $q_0 \leq 0.2s + 1$. $\square$

By Fact 4 and (3), $s' = s - q_0 \geq .8s - 1 \geq s_0$. Since $\mathcal{F}'_{q_0}$ has property ONE and $n' \geq 3s' + 3$, by the assumption of Lemma 2,

$$|\mathcal{F}'_{q_0}| \leq m_{\text{ONE}}(n', s') \leq M(n', s') = M(n - q_0, s - q_0).$$

We are going to show by inverse induction on q that for $q = q_0, \ldots, 0$, $|\mathcal{F}'_q| \leq M(n - q, s - q)$. By estimate (iii) from Fact 2, for $q \geq 1$,

$$n - q \geq n_1(s) - 1 - q \geq n_1(s - q) + 2q - 1 - q \geq n_1(s - q)$$

and, consequently, $M(n - q, s - q) = b(n - q, s - q) = \binom{n-q}{3} - \binom{n-s}{3}$. Thus, the inductive step, for $1 \leq q \leq q_0$, can be easily verified:

$$|\mathcal{F}'_{q-1}| \leq \binom{n - q}{2} + |\mathcal{F}'_q| \leq \binom{n - q}{2} + \binom{n - q}{3} - \binom{n - s}{3}$$

$$= \binom{n-q+1}{3} - \binom{n-s}{3}.$$

The case $q = 0$, that is, the inequality $|\mathcal{F}'_0| \le M(n,s)$, contradicts our assumption that $|\mathcal{F}| = |\mathcal{F}'| = |\mathcal{F}'_0| > M(n.s)$. $\square$

3. Proof of Lemma 1

Before we turn to the actual proof, we need to prove more facts about stable 3-graphs. Let $\mathcal{F}$ be a stable 3-graph with vertex set $[n]$, $n \ge 3s+3$. Set $\nu(\mathcal{F}) = s$ and define

$$\mathcal{F}_0 = \left\{ F \cap [3s+2] : F \in \mathcal{F} \right\}.$$

Since, by stability, there is an s-matching in $\mathcal{F}$ with vertex set $[3s]$, no edge of $\mathcal{F}$ can be disjoint from $[3s]$, and even more so from $[3s+2]$. Hence, $\emptyset \notin \mathcal{F}_0$. Similarly, if, in addition, $\mathcal{F}$ had property ONE, then there would be an s-matching in $\mathcal{F}$ with vertex set $[2, 3s+1]$ and so, no edge of $\mathcal{F}$ might share just one vertex with $[3s+2]$. Hence,

$$\mathcal{F} \in ONE \quad \Rightarrow \quad \forall\, H \in \mathcal{F}_0 :\ |H| \ge 2. \tag{5}$$

The following two observations play a crucial role in the proof of Lemma 1.

FACT 5. *If $\mathcal{F}$ is stable, then $\mathcal{F}_0$ is stable and $\nu(\mathcal{F}_0) = s$.*

PROOF. The stability of $\mathcal{F}_0$ follows directly from the stability of $\mathcal{F}$. Since $\mathcal{F}_0$ contains an s-matching of $\mathcal{F}$, we also have $\nu(\mathcal{F}_0) \ge s$. If $\nu(\mathcal{F}_0) > s$, then there would exist $s+1$ disjoint subsets $H_1, \ldots, H_{s+1}$ of $[3s+2]$, each of size 1, 2 or 3, and $s+1$ triples $F'_1, \ldots, F'_{s+1}$ such that $H_i \subseteq F'_i$, $i = 1, \ldots, s+1$, and $F'_i \cap [3s+2] = H_i$. Let us choose the H_i's and the F'_i's to maximize $|\bigcup_{i=1}^{s+1} F'_i|$. If the union had fewer than $3s+3$ vertices, then some two triples, say F'_1 and F'_2, would intersect, and thus there would exist a vertex $v \in [3s+2] \setminus \bigcup_{i=1}^{s+1} F'_i$. Then, denoting by u a common vertex of F'_1 and F'_2, we could replace F'_1 by $(F'_1 \setminus \{u\}) \cup \{v\}$, obtaining a new family with a larger union, a contradiction. Thus, $F'_1, \ldots, F'_{s+1}$ are pairwise disjoint, which contradicts the assumption that $\nu(\mathcal{F}) = s$. $\square$

We say that $\mathcal{F}$ is *maximal* if for every $E \notin \mathcal{F}$, $\nu(\mathcal{F} \cup \{E\}) > \nu(\mathcal{F})$.

FACT 6. *If $\mathcal{F}$ is stable and maximal, then $\mathcal{F}$ is closed under taking 3-element supersets of the sets of size two in $\mathcal{F}_0$.*

PROOF. Let $H \in \mathcal{F}_0$, $|H| = 2$. There exists $v \notin [3s+2]$ such that $H \cup \{v\} \in \mathcal{F}$. By stability, also $H \cup \{v'\} \in \mathcal{F}$ for all $v' < v$, $v' \notin H$. Suppose that there exists $u > v$ with $H \cup \{u\} \notin \mathcal{F}$. By maximality, the only reason for that is that there is an $(s+1)$-matching $F_1, \ldots, F_{s+1}$ in $\mathcal{F} \cup \{H \cup \{u\}\}$. Without loss of generality set $F_{s+1} = H \cup \{u\}$.

We are going to show that $H \cup \{u\} \notin \mathcal{F}$ implies that one can replace u by some w with $F_1, \ldots, F_s$, $H \cup \{w\}$ forming a matching in $\mathcal{F}$, which is a contradiction. To find such w observe that $|([3s+2] \setminus H) \cup \{v\}| = 3s+1$, and hence there is $w \in ([3s+2] \setminus H) \cup \{v\}$ not belonging to $\bigcup_{i=1}^{s} F_i$. But then, replacing F_{s+1} with $H \cup \{w\}$ leads to an $(s+1)$-matching in $\mathcal{F}$. □

3.1. Set-up. Let integers s and n and a 3-graph $\mathcal{F}$ be such that
(i) $s \geq 25$,
(ii) $n \in \{n_1(s) - 1, n_1(s)\}$,
(iii) $\nu(\mathcal{F}) = s$,
(iv) $\mathcal{F}$ has property ONE (and so, $\mathcal{F}$ is stable),
(v) $|\mathcal{F}| = m_{\mathrm{ONE}}(n, s)$.
In addition, as it is shown below, we may also assume that
(vi) $\mathcal{F}$ is maximal (with respect to $\nu(\mathcal{F}) = s$).

FACT 7. *If $\mathcal{F}$ satisfies* (iii)–(v), *then $\mathcal{F}$ is maximal.*

PROOF. Observe that for each $E \notin \mathcal{F}$,

$$\nu(\mathcal{F} \cup \{E\}) \geq \nu(sh(\mathcal{F} \cup \{E\})) \geq \nu(\mathcal{F}) = s,$$

the second inequality due to the inclusion $\mathcal{F} \subset sh(\mathcal{F} \cup \{E\})$. Suppose that $\nu(\mathcal{F} \cup \{E\}) = \nu(sh(\mathcal{F} \cup \{E\})) = s$. Then, $sh(\mathcal{F} \cup \{E\})$ has ONE, because $\mathcal{F}$ did. But $|sh(\mathcal{F} \cup \{E\})| > |\mathcal{F}| = m_{\mathrm{ONE}}(n, s)$ which is a contradiction. □

By Fact 3, to prove Lemma 1, it is sufficient to show that a 3-graph $\mathcal{F}$ satisfying (i)-(vi) above has at most $M(n, s)$ edges. As an immediate consequence of (vi) and Fact 6, we obtain a pivotal identity:

$$|\mathcal{F}| = \sum_{H \in \mathcal{F}_0} \binom{n - 3s - 2}{3 - |H|}. \tag{6}$$

Recall (5) and let $\mathcal{F}_0^i$, $i = 2, 3$, stand for the subhypergraph of $\mathcal{F}_0$ consisting of all edges of size i. Then, the above identity can be rewritten as

$$|\mathcal{F}| = |\mathcal{F}_0^3| + (n - 3s - 2)|\mathcal{F}_0^2|.$$

We are going to rewrite identity (6) one more time. By property ONE, there is a matching of size s in $\mathcal{F}$ avoiding vertex 1. Thus, by stability, there is also an s-matching contained in $[3s+2] \setminus \{1\}$. Let $F_1, \ldots, F_s$ form such a matching and let

$$F_0 := \{1, d\} = [3s+2] \setminus (F_1 \cup \cdots \cup F_s)$$

contain the two remaining elements of $[3s+2]$. Among all possible candidates for $F_1, \ldots, F_s$ we choose one which makes d as small as possible. Observe that $F_0 \notin \mathcal{F}_0$, and consequently, for every $v \in [3s+2]$, $v \neq d$, we also have $\{d, v\} \notin \mathcal{F}_0$.

For $H \in \mathcal{F}_0$, we define two parameters: *the spread*

$$z(H) = \big|\big\{\, i \in [s] : H \cap F_i \neq \emptyset \big\}\big|$$

and *the weight*

$$w(H) = \frac{\binom{n-3s-2}{3-|H|}}{\binom{s-z(H)}{3-z(H)}}\,.$$

For each triple of indices $\tau \in \binom{[s]}{3}$, let

$$V^\tau = F_0 \cup \bigcup_{i \in \tau} F_i \quad \text{and} \quad \mathcal{F}_0^\tau = \big\{\, H \in \mathcal{F}_0 : H \subset V^\tau \big\}\,.$$

It follows from (6) and the definition of $w(H)$ that

$$|\mathcal{F}| = \sum_{\tau \in \binom{[s]}{3}} \sum_{H \in \mathcal{F}_0^\tau} w(H)\,. \tag{7}$$

As $M(n,s) \geq a(s) = |\mathcal{A}|$, our ultimate goal is to show that $|\mathcal{F}| \leq |\mathcal{A}|$. Recall that $\mathcal{A}$ is the complete 3-graph on $[3s+2]$. Identity (7) holds also for $\mathcal{A}$ instead of $\mathcal{F}$ (with the same choice of an s-matching $F_1, \ldots, F_s$), but due to the symmetry of $\mathcal{A}$, the inner sums in (7) are independent of the choice of τ, and thus all equal to each other. Denoting this common value by W, we thus have $|\mathcal{A}| = a(s) = \binom{s}{3} W$, and we will achieve our goal by showing that for each $\tau \in \binom{[s]}{3}$

$$\sum_{H \in \mathcal{F}_0^\tau} w(H) \leq \sum_{H \in \mathcal{A}_0^\tau} w(H). \tag{8}$$

3.2. The 11-vertex board. Let us fix $\tau \in \binom{[s]}{3}$. Without loss of generality assume that $\tau = \{1,2,3\}$ and, thus, $V^\tau = F_0 \cup F_1 \cup F_2 \cup F_3$. Set $F_i = \{a_i < b_i < c_i\}$, $i = 1,2,3$, and $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2, c_3\}$ (see Fig. 1). Note that we have reduced our original problem to a new, 'bounded size' problem on an 11-vertex hypergraph $\mathcal{F}_0^\tau$.

The family $\mathcal{F}_0^\tau$ consists of triples and pairs. According to their width we divide the triples into *wide* ($z(H) = 3$), *medium* ($z(H) = 2$), and *narrow* ($z(H) = 1$), with the corresponding weights $w(H)$ equal to

$$1, \quad \frac{1}{s-2}, \quad \text{and} \quad \frac{1}{\binom{s-1}{2}}\,.$$

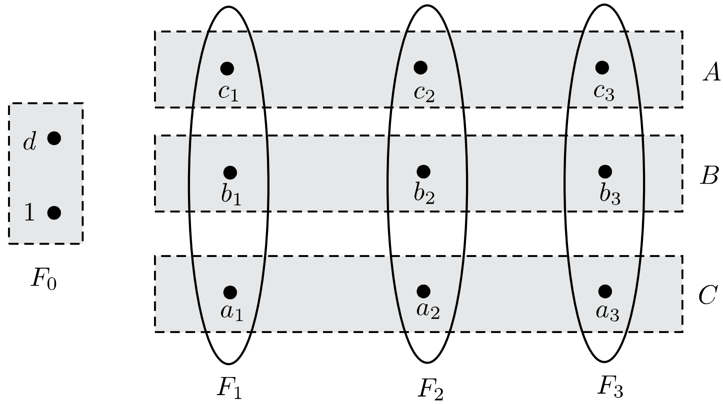

Fig. 1: The 11-vertex board V^τ

Similarly, we divide the pairs into *wide* ($z(H) = 2$) and *narrow* ($z(H) = 1$), with weights

$$\frac{n-3s-2}{s-2} \quad \text{and} \quad \frac{n-3s-2}{\binom{s-1}{2}}\,.$$

Our strategy for proving (8) is as follows. First note that for each triple $H \in \mathcal{F}_0$ its weight $w(H)$ is the same in $\mathcal{F}$ and in $\mathcal{A}$. Consequently, as $\mathcal{A} = \binom{[3s+2]}{3}$, for every triple $H \in \mathcal{F}_0^\tau$ its weights on both sides of (8) cancel out. We will consider several cases with respect to the structure of $\mathcal{F}_0^\tau$, and in each case will argue that the weights $w(H)$ of the pairs $H \in \mathcal{F}_0^\tau$ sum up to no more than the total weight of the triples which are absent from $\mathcal{F}_0^\tau$, therefore establishing (8). Note that for a wide pair H, owing to the estimate $n \le n_1(s) \le 3.5s + 3$ (see Fact 2(i)), we can bound its weight by

$$w(H) = \frac{n-3s-2}{s-2} \le \frac{s/2+1}{s-2}\,.$$

Hence, for large s, it is enough to show that the number of wide pairs present in $\mathcal{F}_0^\tau$ is strictly less than twice the number of wide triples missing from $\mathcal{F}_0^\tau$. For smaller s, however, we need also look at other types of sets (not just wide).

The absence of specified sets from $\mathcal{F}_0^\tau$ will be often forced by the same kind of argument: assuming their presence we would get a matching of size 4 in $\mathcal{F}_0^\tau$, leading to a contradiction with Fact 5. Not to repeat ourselves, we will refer to this argument as *a 4-matching argument*.

As a first example of this technique, consider a narrow pair H which, say, is contained in $F_0 \cup F_1$. If its complement $H' = (F_0 \cup F_1) \setminus H$, which is a narrow triple, belonged to $\mathcal{F}_0^\tau$, then H, H', F_2, and F_3 would form a 4-matching. Thus, $H' \notin \mathcal{F}_0^\tau$, and when proving (8), we can use the bound

$$w(H) - w(H') \le \frac{n-3s-3}{\binom{s-1}{2}} \le \frac{s}{(s-1)(s-2)}, \tag{9}$$

where the second inequality follows again from Fact 2(i).

The next fact sets an upper bound on the total number of narrow pairs in $\mathcal{F}_0^\tau$.

Fact 8. *For each $i \in \{1,2,3\}$, there are at most 3 narrow pairs in $F_0 \cup F_i$.*

Proof. If $\{a_i, c_i\} \in \mathcal{F}_0^\tau$ then, by stability, also $\{1, b_i\} \in \mathcal{F}_0^\tau$, which is a contradiction by the 4-matching argument. By stability again, this also excludes $\{b_i, c_i\}$ which majorizes $\{a_i, c_i\}$. Similarly, we cannot have both, $\{1, c_i\} \in \mathcal{F}_0^\tau$ and $\{a_i, b_i\} \in \mathcal{F}_0^\tau$. Finally, recall that no pair in $\mathcal{F}_0^\tau$ contains d. □

Now we prove the existence of narrow triples which often help to complete a 4-matching in $\mathcal{F}_0^\tau$.

Fact 9. *For each $i \in \{1,2,3\}$, we have $\{1, d, b_i\} \in \mathcal{F}_0^\tau$ (and thus, by stability, also $\{1, d, a_i\} \in \mathcal{F}_0^\tau$).*

Proof. If $b_i > d$, then $\{1, d, b_i\} \prec \{a_i, b_i, c_i\} = F_i \in \mathcal{F}_0^\tau$, and so, by stability, $\{1, d, b_i\} \in \mathcal{F}_0^\tau$. Assume now that $b_i < d$. We claim that $\{1, b_i\} \in \mathcal{F}_0$. Indeed, otherwise, by maximality of $\mathcal{F}$, there is in $\mathcal{F}$ an s-matching disjoint from $\{1, b_i\}$ which, by stability, implies the presence of such a matching in $[3s+2] \setminus \{1, b_i\}$. This contradicts the minimality of d. Hence, $\{1, b_i\} \in \mathcal{F}_0$ and, by Fact 6, $\{1, b_i, d\} \in \mathcal{F}$. □

3.3. The proof of (8). It is split into two major cases, quite uneven in length.

Case I: There exist $F', F'' \in \mathcal{F}_0^\tau$ such that $F' \cup F'' = B \cup C$.

We claim there is no wide pair in $\mathcal{F}_0^\tau$. Indeed, otherwise, by stability, there would be a pair $H \in \mathcal{F}_0^\tau$ with $H \subset A$. Without loss of generality assume that $a_1 \notin H$. Then F', F'', H, and $\{1, d, a_1\}$ (by Fact 9) would form a 4-matching in $\mathcal{F}_0^\tau$, a contradiction. In view of the absence of wide pairs, our goal is to outweigh the contribution (to the left-hand side of (8)) of the narrow pairs by the contribution (to the right-hand side of (8)) of the narrow and medium triples which are missing from $\mathcal{F}_0^\tau$. The narrow triples are already taken care of via estimate (9). If there is a narrow pair in $\mathcal{F}_0^\tau$ which is contained in, say, $F_0 \cup F_1$, then by stability, also $\{1, a_1\} \in \mathcal{F}_0^\tau$. Consequently, a 4-matching argument, like the one used above, excludes from $\mathcal{F}_0^\tau$

the medium triple $\{d, a_2, a_3\}$ along with all 8 of its majorants. Thus, (8) holds if

$$\frac{3s}{(s-1)(s-2)} \leq \frac{9}{s-2},$$

which is true for $s \geq 3$.

Case II: There are no sets $F', F'' \in \mathcal{F}_0^\tau$ such that $F' \cup F'' = B \cup C$.

Note that if at least 5 of all 8 wide triples contained in $B \cup C$ were present in $\mathcal{F}_0^\tau$, then Case I would hold. Thus, there are at least 4 wide triples missing from $\mathcal{F}_0^\tau$ (and contained in $B \cup C$). By Fact 8 we have always at most 9 narrow pairs and we will not attempt to refine this bound. Instead, in each case we will try to bound from above the numbers of wide pairs and, from below, the number of wide triples, by, respectively, x and y, so that

$$x \times \frac{s/2+1}{s-2} + \frac{9s}{(s-1)(s-2)} \leq y. \tag{10}$$

In conclusion, if for some s, our bounds x and y satisfy (10), then (8) holds.

Case II will have several subcases, for concise description of which we introduce the following notation:

$$BC = \left\{\text{wide pairs } H \in \mathcal{F}_0^\tau : H \cap B \neq \emptyset,\ H \cap C \neq \emptyset\right\},$$

$$\overline{BC} = BC \cup \left(\binom{C}{2} \cap \mathcal{F}_0^\tau\right), \quad \text{and} \quad \underline{\overline{BC}} = \overline{BC} \cup \left(\binom{B}{2} \cap \mathcal{F}_0^\tau\right).$$

Sets (of wide pairs) AB, AC, $\overline{AB}$, and $\underline{\overline{AB}}$ are defined analogously.

In each subcase below (except for the last one) we just derive the bounds x and y and claim that (10) holds (for $s \geq 25$), leaving numerical details to the untrusting reader. Recall that throughout Case II we can always take $y = 4$, though we can often do better. Also, when applying the 4-matching argument, we will be often employing Fact 9.

Subcase II.1: $\underline{\overline{BC}} \neq \emptyset$. Without loss of generality, let $\{c_1, b_2\} \in \mathcal{F}_0^\tau$. But then $\{c_1, b_2\}$, together with $\{a_1, b_1\}$, $\{1, d, b_1\}$, and F_3, forms a 4-matching in $\mathcal{F}_0^\tau$, a contradiction.

Subcase II.2: $\underline{\overline{BC}} = \emptyset$. This subcase is further subdivided according to the size and structure of the graph AC. Note that if $|AC| > 2$ then AC must have two disjoint edges.

Subcase II.2.(i): there is a 2-matching M in AC. Say $a_1, a_2 \in V(M)$. Then, by the 4-matching argument, $\{b_1, b_2, a_3\} \notin \mathcal{F}_0^\tau$, and so, $y = 12$. As we also have $x \leq 18$, (10) holds. (By Fact 6, we can improve the upper bound on the number of wide pairs to $x = 27 - (4+6+6) = 11$, but we do not need it.)

Subcase II.2.(ii): $|AC| = 2$ and the two edges in AC have a common endpoint in C, say $\{a_1, c_2\}, \{a_3, c_2\} \in \mathcal{F}_0^\tau$. Then, by stability, also $\{a_3, a_2\} \in \mathcal{F}_0^\tau$, which, by a 4-matching argument, excludes both $\{c_1, b_2, b_3\}$ and $\{b_1, b_2, c_3\}$ from $\mathcal{F}_0^\tau$. This sets $y = 6$. As for x, note that by a similar 4-matching argument, we may exclude from $\mathcal{F}_0^\tau$ both $\{b_1, a_2\}$ and $\{b_3, a_2\}$, and thus, by stability also $\{b_1, b_2\}$ and $\{b_2, b_3\}$. Hence, $|\overline{AB}| \leq 8$ and so $x = 10$ which yields (10).

Subcase II.2.(iii): $|AC| = 2$ and the two edges in AC have a common endpoint in A, say $\{c_1, a_2\}, \{c_3, a_2\} \in \mathcal{F}_0^\tau$. We claim that $|\overline{AB}| \leq 8$. Indeed, by a 4-matching argument, either $\{a_1, b_3\}$ or $\{b_1, a_3\}$ is not in $\mathcal{F}_0^\tau$. By symmetry, assume that $\{a_1, b_3\} \notin \mathcal{F}_0^\tau$. By a similar argument, $\{a_1, b_2\}$ or $\{b_1, a_3\}$ is not in $\mathcal{F}_0^\tau$, and $\{b_1, b_3\}$ or $\{a_1, a_3\}$ is not in $\mathcal{F}_0^\tau$. In addition, $\{b_1, b_3\} \notin \mathcal{F}_0^\tau$, by stability. This yields at least 4 wide pairs missing from $\overline{AB}$, and thus, $|\overline{AB}| \leq 12 - 4 = 8$.

If any of $\{b_1, c_2, a_3\}, \{a_1, c_2, b_3\}$, or $\{c_1, b_2, a_3\}$ is missing from $\mathcal{F}_0^\tau$, then, by stability, at least 6 wide triples are not in $\mathcal{F}_0^\tau$. Thus, we have $x = 10$, $y = 6$, and (10) follows. If, on the other hand, all these three triples are present in $\mathcal{F}_0^\tau$, then a 4-matching argument and stability imply that $\overline{AB} = \emptyset$. Thus, we have $x = 5$ and $y = 4$, and (10) holds again.

Subcase II.2.(iv): $|AC| = 1$, say $\{a_1, c_2\} \in \mathcal{F}_0^\tau$. If either of $\{c_1, a_2, b_3\}$ and $\{c_1, b_2, a_3\}$ is missing from $\mathcal{F}_0^\tau$, then $y = 6$ and, by a 4-matching argument, at least 3 pairs from $\overline{AB}$ are missing ($\{b_1, b_2\}$ or $\{a_2, b_3\}$, $\{b_1, b_3\}$ or $\{b_2, a_3\}$, and $\{b_2, b_3\}$ or $\{a_2, a_3\}$). Thus, $x = 10$ and (10) follows. Otherwise, by a 4-matching argument, $\overline{AB} \subseteqq \binom{A}{2} \cup \{\{a_1, b_2\}, \{a_1, b_3\}\}$. For instance, if $\{b_1, a_2\} \in \overline{AB}$, then $\{b_1, a_2\}$, together with $\{a_1, c_2\}$, $\{c_1, b_2, a_3\}$, and $\{1, d, b_3\}$ (see Fact 9), would form a 4-matching in $\mathcal{F}_0^\tau$, a contradiction. Hence, $x = 6$ (and $y = 4$), and (10) holds again.

Subcase II.2.(v): $AC = \emptyset$. In this case, the total number of wide pairs coincides with $|\overline{AB}|$ and we consider several subcases with respect to this quantity.

$|\overline{AB}| \leq 6$. In this case, $x = 6$, $y = 4$, and (10) holds.

$|\overline{AB}| = 7$. It can be easily checked by inspection that $\overline{AB}$ contains a matching M of size 2 such that $A \setminus V(M) \neq \emptyset$, say $a_1 \notin V(M)$. Then, by a 4-matching argument, $\{a_1, c_2, c_3\} \notin \mathcal{F}_0^\tau$. Since, in addition, there are at least 4 wide triple in $B \cup C$ missing from $\mathcal{F}_0^\tau$, we have $y = 5$, $x = 7$, and (10) holds again.

$8 \leq |\overline{AB}| \leq 10$. Eight edges in $\overline{AB}$ guarantee two matchings of size 2 in $\overline{AB}$, M' and M'', and two distinct vertices $a', a'' \in A$ such that $a' \notin V(M')$ and $a'' \notin V(M'')$. Let $a' = a_1$ and $a'' = a_2$. Then, by the 4-matching argument, $\{a_1, c_2, c_3\} \notin \mathcal{F}_0^\tau$ and $\{c_1, a_2, c_3\} \notin \mathcal{F}_0^\tau$. Hence, $x = 10$, $y = 6$, and (10) follows.

$|\overline{AB}| = 11$. In this case there are three matchings of size 2 in $\overline{AB}$, M_1, M_2, M_3, such that $a_i \notin V(M_i)$, $i = 1, 2, 3$, and (10) follows with $x = 11$ and $y = 7$.

$|\overline{AB}| = 12$. If we just repeated the argument from the previous subcase, we would have $x = 12$ and $y = 7$, and (10) would hold for $s \geq 33$ only. To push it down to $s \geq 25$, we need to refine our argument and turn for help to medium triples. But this is easy: all 12 triples of the form $\{a_i, c_i, c_j\}$ or $\{b_i, c_i, c_j\}$, $i \neq j$, are forbidden in $\mathcal{F}_0^\tau$. Indeed, if, say, $\{a_1, c_1, c_2\} \in \mathcal{F}_0^\tau$, then we would get a 4-matching consisting of $\{a_1, c_1, c_2\}$, $\{a_2, b_3\}$, $\{b_2, a_3\}$, and $\{1, d, b_1\}$. Thus, (8) follows from

$$12 \times \frac{s/2 + 1}{s - 2} + \frac{9s}{(s-1)(s-2)} - \frac{12}{s-2} \leq 7,$$

which is true for $s \geq 25$. The proof of Lemma 1, and thus the proof of Theorem 1, is completed.

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