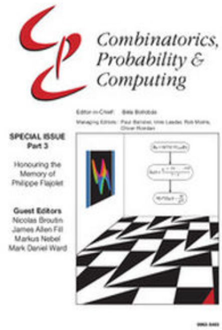
Combinatorics, Probability and Computing

http://journals.cambridge.org/CPC



Loose Hamilton Cycles in Regular Hypergraphs

ANDRZEJ DUDEK, ALAN FRIEZE, ANDRZEJ RUCIŃSKI and MATAS ŠILEIKIS

Combinatorics, Probability and Computing / Volume 24 / Special Issue 01 / January 2015, pp 179 - 194
DOI: 10.1017/S0963548314000406, Published online: 24 September 2014

Combinatorics, Probability and Computing (2015) 24, 179–194. © Cambridge University Press 2014
doi:10.1017/S0963548314000406

Loose Hamilton Cycles in Regular Hypergraphs

ANDRZEJ DUDEK[1]†, ALAN FRIEZE[2]‡, ANDRZEJ RUCIŃSKI[3]§
and MATAS ŠILEIKIS[4]¶

[1] Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA
(e-mail: `andrzej.dudek@wmich.edu`)

[2] Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA
(e-mail: `alan@random.math.cmu.edu`)

[3] Department of Discrete Mathematics, Adam Mickiewicz University, 61-614 Poznań, Poland
(e-mail: `rucinski@amu.edu.pl`)

[4] Department of Mathematics, Uppsala University, Box 480, 751 06 Uppsala, Sweden
(e-mail: `matas.sileikis@math.uu.se`)

Received 15 March 2013; revised 1 August 2013; first published online 24 September 2014

We establish a relation between two uniform models of random k-graphs (for constant $k \geqslant 3$) on n labelled vertices: $\mathbb{H}^{(k)}(n,m)$, the random k-graph with exactly m edges, and $\mathbb{H}^{(k)}(n,d)$, the random d-regular k-graph. By extending the switching technique of McKay and Wormald to k-graphs, we show that, for some range of $d = d(n)$ and a constant $c > 0$, if $m \sim cnd$, then one can couple $\mathbb{H}^{(k)}(n,m)$ and $\mathbb{H}^{(k)}(n,d)$ so that the latter contains the former with probability tending to one as $n \to \infty$. In view of known results on the existence of a loose Hamilton cycle in $\mathbb{H}^{(k)}(n,m)$, we conclude that $\mathbb{H}^{(k)}(n,d)$ contains a loose Hamilton cycle when $d \gg \log n$ (or just $d \geqslant C \log n$, if $k = 3$) and $d = o(n^{1/2})$.

2010 *Mathematics subject classification*: Primary 05C65, 05C80
Secondary 05C38

1. Introduction

A k-*uniform hypergraph* (or k-*graph* for short) on a vertex set $V = \{1,\ldots,n\}$ is a family of k-element subsets of V. A k-graph $H = (V,E)$ is d-regular if the degree of every vertex

† Supported in part by Simons Foundation grant #244712.
‡ Supported in part by NSF grant CCF1013110.
§ Research supported by the Polish NSC grant N201 604940. Part of the research was performed at Emory University, Atlanta.
¶ Research supported by the Polish NSC grant N201 604940. Part of the research was performed at Adam Mickiewicz University, Poznań.

is d:

$$\deg(v) := |\{e \in E : v \in e\}| = d, \quad v = 1, \ldots, n.$$

Let $\mathcal{H}^{(k)}(n,d)$ be the family of all such graphs. Further we tacitly assume that k divides nd. By $\mathbb{H}^{(k)}(n,d)$ we denote the *regular* random graph, which is chosen uniformly at random from $\mathcal{H}^{(k)}(n,d)$. Let

$$M := nd/k$$

stand for the number of edges of $\mathbb{H}^{(k)}(n,d)$.

Let us recall two more standard models of random k-graphs on n vertices. For $p \in [0,1]$, the *binomial* random k-graph $\mathbb{H}^{(k)}(n,p)$ is a random k-graph obtained by including each of the $\binom{n}{k}$ possible edges with probability p independently of the others. For integer $m \in [0, \binom{n}{k}]$, the *uniform* random graph $\mathbb{H}^{(k)}(n,m)$ is chosen uniformly at random among k-graphs with precisely m edges.

We study the behaviour of random k-graphs as $n \to \infty$. Parameters d, m, p are treated as functions of n. We use the asymptotic notation $O(\cdot), o(\cdot), \Theta(\cdot), \sim$ (as it is defined in, say, [15]), with respect to n tending to infinity and assume that implied constants may depend on k. Given a sequence of events $(\mathcal{A}_n)$, we say that $\mathcal{A}_n$ happens *asymptotically almost surely* (a.a.s.) if $\mathbb{P}(\mathcal{A}_n) \to 1$, as $n \to \infty$.

The main result of the paper is that we can couple $\mathbb{H}^{(k)}(n,d)$ and $\mathbb{H}^{(k)}(n,m)$ so that the latter is a subgraph of the former a.a.s.

Theorem 1.1. *For every $k \geqslant 3$, there are positive constants c and C such that if $d \geqslant C \log n$, $d = o(n^{1/2})$ and $m = \lfloor cM \rfloor = \lfloor cnd/k \rfloor$, then one can define a joint distribution of random graphs $\mathbb{H}^{(k)}(n,d)$ and $\mathbb{H}^{(k)}(n,m)$ in such a way that*

$$\mathbb{H}^{(k)}(n,m) \subset \mathbb{H}^{(k)}(n,d) \quad \textit{a.a.s.}$$

To prove Theorem 1.1, we consider a generalization of a k-graph that allows loops and multiple edges. By a k-*multigraph* on the vertex set $[n]$ we mean a multiset of k-element multisubsets of $[n]$. An edge is called a *loop* if it contains more than one copy of some vertex and otherwise it is called a *proper edge*.

The idea of the proof and the structure of the paper are as follows. In Section 2 we generate two models of random k-multigraphs by drawing random sequences from $[n]$ and cutting them into consecutive segments of length k. By accepting an edge only if it is not a loop and does not coincide with a previously accepted edge, after m successful trials we obtain $\mathbb{H}^{(k)}(n,m)$. On the other hand, by allowing d copies of each vertex, and accepting every edge, after dn/k steps we obtain a d-regular k-multigraph $\mathbb{H}_*^{(k)}(n,d)$. Then we show that $\mathbb{H}_*^{(k)}(n,d)$ a.a.s. has no multiple edges and relatively few loops. In Section 3 we couple the two random processes in such a way that $\mathbb{H}^{(k)}(n,m)$ is a.a.s. contained in an initial segment of $\mathbb{H}_*^{(k)}(n,d)$, which we call *red*. In Section 4 we eliminate at once all red loops of $\mathbb{H}_*^{(k)}(n,d)$ by swapping them with randomly selected non-red (green) proper edges. Finally, in Section 5, we eliminate the green loops one by one using a certain random procedure (called switching) which does not destroy the previously embedded copy of

$\mathbb{H}^{(k)}(n,m)$ and, at the same time, transforms $\mathbb{H}_*^{(k)}(n,d)$ into a k-graph $\tilde{\mathbb{H}}^{(k)}(n,d)$, which is distributed approximately as $\mathbb{H}^{(k)}(n,d)$, that is, almost uniformly. Theorem 1.1 follows by a (maximal) coupling of $\tilde{\mathbb{H}}^{(k)}(n,d)$ and $\mathbb{H}^{(k)}(n,d)$.

A consequence of Theorem 1.1 is that $\mathbb{H}^{(k)}(n,d)$ inherits from $\mathbb{H}^{(k)}(n,m)$ properties that are increasing, that is to say, properties that are preserved as new edges are added. An example of such a property is Hamiltonicity, that is, containment of a Hamilton cycle.

A *loose Hamilton cycle* on n vertices is a set of edges $e_1,\dots,e_l$ such that, for some cyclic order of the vertices, every edge e_i consists of k consecutive vertices, $|e_i \cap e_{i+1}| = 1$ for every $i \in [l]$, where $e_{l+1} := e_1$. A necessary condition for the existence of a loose Hamilton cycle on n vertices is $(k-1)|n$, which we will assume whenever relevant.

The history of Hamiltonicity of regular graphs is rich and exciting (see [21]). However, we state only the final results here. Asymptotic Hamiltonicity was proved by Robinson and Wormald [20] in 1994 for any fixed $d \geqslant 3$, by Krivelevich, Sudakov, Vu and Wormald [16] in 2001 for $d \geqslant n^{1/2}\log n$, and by Cooper, Frieze and Reed [7] in 2002 for $C \leqslant d \leqslant n/C$ and some large constant C.

The threshold for existence of a loose Hamilton cycle in $\mathbb{H}^{(k)}(n,p)$ was determined by Frieze [12] (for $k=3$) as well as Dudek and Frieze [8] (for $k \geqslant 4$) under a divisibility condition $2(k-1)|n$, which was relaxed to $(k-1)|n$ by Dudek, Frieze, Loh and Speiss [10].

However, we formulate these results for the model $\mathbb{H}^{(k)}(n,m)$, such a possibility being provided to us by the asymptotic equivalence of models $\mathbb{H}^{(k)}(n,p)$ and $\mathbb{H}^{(k)}(n,m)$ (see, *e.g.*, Corollary 1.16 in [13]).

Theorem 1.2 ([12], [10]). *There is a constant $C>0$ such that if $m \geqslant Cn\log n$, then*

$$\mathbb{H}^{(3)}(n,m) \text{ contains a loose Hamilton cycle a.a.s.}$$

Theorem 1.3 ([8], [10]). *Let $k \geqslant 4$. If $n\log n = o(m)$, then*

$$\mathbb{H}^{(k)}(n,m) \text{ contains a loose Hamilton cycle a.a.s.}$$

Theorems 1.1, 1.2, and 1.3 immediately imply another main result of this paper.

Theorem 1.4. *There is a constant $C>0$ such that if $C\log n \leqslant d = o(n^{1/2})$, then*

$$\mathbb{H}^{(3)}(n,d) \text{ contains a loose Hamilton cycle a.a.s.}$$

For every $k \geqslant 4$ if $\log n = o(d)$ and $d = o(n^{1/2})$, then

$$\mathbb{H}^{(k)}(n,d) \text{ contains a loose Hamilton cycle a.a.s.}$$

2. Preliminaries

We say that a k-multigraph is *simple* if it is a k-graph, that is, if it contains neither multiple edges nor loops.

Given a sequence $\mathbf{x} \in [n]^{ks}$, $s \in \{1, 2, \dots\}$, let $\mathsf{H}(\mathbf{x})$ stand for a k-multigraph with s edges

$$x_{ki+1} \dots x_{ki+k}, \quad i = 0, \dots, s-1.$$

In what follows it will be convenient to work directly with the sequence $\mathbf{x}$ rather than with the k-multigraph $\mathsf{H}(\mathbf{x})$. Recycling the notation, we still refer to the k-tuples of $\mathbf{x}$ which correspond to the edges, loops, and proper edges of $\mathsf{H}(\mathbf{x})$ as *edges*, *loops*, and *proper edges* of $\mathbf{x}$, respectively. We say that $\mathbf{x}$ contains *multiple edges* if $\mathsf{H}(\mathbf{x})$ contains multiple edges, that is, some two edges of $\mathbf{x}$ are identical as multisets. By $\lambda(\mathbf{x})$ we denote the number of loops in $\mathbf{x}$.

Let $\mathbf{X} = (X_1, \dots, X_{nd})$ be a sequence of i.i.d. random variables, each distributed uniformly over $[n]$:

$$\mathbb{P}(X_i = j) = \frac{1}{n}, \quad 1 \leqslant i \leqslant nd, \quad 1 \leqslant j \leqslant n.$$

Set

$$L := n^{1/4} d^{1/2}.$$

Proposition 2.1. *If $d \to \infty$ and $d = o(n^{1/2})$, then a.a.s. $\mathbf{X}$ has no multiple edges and $\lambda(\mathbf{X}) \leqslant L$.*

Proof. Both statements hold a.a.s. by Markov's inequality, because the expected number of pairs of multiple edges in $\mathbf{X}$ is at most

$$\binom{M}{2} \frac{k!}{n^k} = O(d^2 n^{2-k}) = o(1),$$

and the expected number of loops in $\mathbf{X}$ is

$$\mathbb{E}\lambda(\mathbf{X}) \leqslant M \binom{k}{2} n^{-1} = O(d) = o(n^{1/4} d^{1/2}). \qquad \square$$

Let $\mathcal{S} \subset [n]^{nd}$ be the family of all sequences in which every value $i \in [n]$ occurs precisely d times. Let $\mathbf{Y} = (Y_1, \dots, Y_{nd})$ be a sequence chosen from $\mathcal{S}$ uniformly at random. One can equivalently define $\mathbf{Y}$ as a discrete time process determined by the conditional probabilities

$$\mathbb{P}\big(Y_{t+1} = v \mid Y_1, \dots, Y_t\big) = \frac{d - \deg_t(v)}{nd - t}, \quad v = 1, \dots, n, \quad t = 0, \dots, nd-1, \tag{2.1}$$

where

$$\deg_t(v) := |\{1 \leqslant s \leqslant t : Y_s = v\}|.$$

Assuming $k|(nd)$, we define a random d-regular k-multigraph

$$\mathbb{H}_*^{(k)}(n, d) := \mathsf{H}(\mathbf{Y}).$$

Note that for every $H \in \mathcal{H}^{(k)}(n, d)$ the number of sequences giving H is the same, namely, $M!(k!)^M$. Therefore $\mathbb{H}^{(k)}(n, d)$ can be obtained from $\mathbb{H}_*^{(k)}(n, d)$ by conditioning on simplicity.

Probably a more popular way to define $\mathbb{H}_*^{(k)}(n, d)$ is via the so-called *configuration model*, which, for $k = 2$, first appeared implicitly in Bender and Canfield [2] and was given in its explicit form by Bollobás [3] (its generalization to every k is straightforward). A *configuration* is a partition of the set $[n] \times [d]$ into M sets of size k, say, $P_1, \ldots, P_M$. Then $\mathbb{H}_*^{(k)}(n,d)$ is obtained by taking a configuration uniformly at random and mapping every set $P_i = \{(v_1, w_1), \ldots, (v_k, w_k)\}$ to an edge $v_1 \ldots v_k$.

The idea of obtaining $\mathcal{H}^{(k)}(n,d)$ from a random sequence for $k = 2$ was used independently by Bollobás and Frieze [5] and Chvátal [6].

What makes studying d-regular k-graphs a bit easier than graphs, at least for small d, is that a.a.s. $\mathbf{Y}$ has no multiple edges. However, they usually have a few loops, though, as it turns out, not too many. Throughout the paper, for $r = 0, 1, \ldots$ and $x \in \mathbb{R}$, we use the standard notation $(x)_r := x(x-1)\ldots(x-r+1)$. Recall that $L = n^{1/4}d^{1/2}$.

Proposition 2.2. *If $d \to \infty$, and $d = o(n^{1/2})$, then each of the following statements holds a.a.s.:*

(i) $\mathbf{Y}$ *has no multiple edges,*
(ii) $\mathbf{Y}$ *has no loop with a vertex of multiplicity at least* 3*,*
(iii) $\mathbf{Y}$ *has no loop with two vertices of multiplicity at least* 2*,*
(iv) $\lambda(\mathbf{Y}) \leqslant L$.

Proof. The first three statements hold because the expected number of undesired objects tends to zero.

(i) The expected number of pairs of multiple edges in $\mathbf{Y}$ is

$$\binom{M}{2} \sum_{k_1+\ldots+k_n=k} \frac{\binom{k}{k_1,\ldots,k_n}^2 \binom{nd-2k}{d-2k_1,\ldots,d-2k_n}}{\binom{nd}{d,\ldots,d}} \leqslant n^2d^2n^k \frac{k!^2d^{2k}}{(nd)_{2k}} = O(n^{2-k}d^2) = o(1).$$

(ii) The expected number of loops in $\mathbf{Y}$ having a vertex of multiplicity at least 3 is at most

$$M \frac{\binom{k}{3} n \binom{nd-3}{d-3,d,\ldots,d}}{\binom{nd}{d,\ldots,d}} \leqslant nd \frac{k^3nd^3}{(nd)_3} = O(n^{-1}d) = o(1).$$

(iii) Similarly the expected number of loops in $\mathbf{Y}$ having at least two vertices of multiplicity at least 2 is at most

$$M \frac{\binom{k}{2}\binom{k-2}{2} n^2 \binom{nd-4}{d-2,d-2,d,\ldots,d}}{\binom{nd}{d,\ldots,d}} \leqslant nd \frac{k^4n^2d^4}{(nd)_4} = O(n^{-1}d) = o(1).$$

(iv) This follows from the Markov inequality, since

$$\mathbb{E}\lambda(\mathbf{Y}) \leqslant M \frac{\binom{k}{2} n \binom{nd-2}{d-2,d,\ldots,d}}{\binom{nd}{d,\ldots,d}} \leqslant nd \frac{k^2nd^2}{(nd)_2} = O(d) = o(n^{1/4}d^{1/2}). \qquad \square$$

In a couple of forthcoming proofs we will need the following concentration inequality (see, *e.g.*, McDiarmid [17, §3.2]). Let $\mathcal{S}_N$ be the set of permutations of $[N]$ and let $\mathbf{Z}$ be distributed uniformly over $\mathcal{S}_N$. Suppose that function $f : \mathcal{S}_N \to \mathbb{R}$ satisfies a Lipschitz property, that is, for some $b > 0$,

$$|f(\mathbf{z}) - f(\mathbf{z}')| \leqslant b,$$

whenever $\mathbf{z}'$ can be obtained from $\mathbf{z}$ by swapping two elements. Then

$$\mathbb{P}\big(|f(\mathbf{Z}) - \mathbb{E}f(\mathbf{Z})| \geqslant t\big) \leqslant 2\mathrm{e}^{-2t^2/b^2N}, \quad t \geqslant 0. \tag{2.2}$$

We set $r := 2^k + 1$ and $c := 1/(2r+1)$. For the rest of the paper let

$$m := \lfloor cM \rfloor.$$

Colour the first rm edges of $\mathbf{Y}$ *red* and the remaining $M - rm$ edges *green*. Define $\mathbf{Y}_{\text{red}} = (Y_1, \dots, Y_{krm})$ and $\mathbf{Y}_{\text{green}} = (Y_{krm+1}, \dots, Y_{nd})$. Consider a function $\varphi : \mathcal{S} \to \mathbb{Z}$ defined by

$$\varphi(\mathbf{y}) := \sum_{v=1}^{n} (\deg_{\text{green}}(\mathbf{y}; v))_2,$$

where $\deg_{\text{green}}(\mathbf{y}; v) := |\{i \in [rkm+1, kM] : y_i = v\}|$ is the *green degree* of v. It can be easily checked that

$$\mathbb{E}\varphi(\mathbf{Y}) = n(d)_2 \frac{(kM - rkm)_2}{(kM)_2} = \Theta(nd^2). \tag{2.3}$$

Suppose that sequences $\mathbf{y}, \mathbf{z} \in \mathcal{S}$ can be obtained from each other by swapping two coordinates. Since such a swapping affects the green degree of at most two vertices and for every such vertex the green degree changes by at most one, we get

$$|\varphi(\mathbf{y}) - \varphi(\mathbf{z})| \leqslant 2 \max_{1 \leqslant r \leqslant d} \{(r)_2 - (r-1)_2\} = 2((d)_2 - (d-1)_2) < 4d.$$

Thus, treating $\mathbf{Y}$ as a permutation of nd elements, (2.2) implies

$$\mathbb{P}(|\varphi(\mathbf{Y}) - \mathbb{E}\varphi(\mathbf{Y})| \geqslant x) \leqslant 2\exp\left\{-\frac{x^2}{8nd^3}\right\}, \quad x > 0. \tag{2.4}$$

3. Embedding $\mathbb{H}^{(k)}(n, m)$ into $\mathbb{H}_*^{(k)}(n, d)$

A crucial step towards the embedding is to couple the processes (X_t) and (Y_t), $t = 1, \dots, nd$, in such a way that a.a.s. $\mathbf{X}$ and $\mathbf{Y}$ have many edges in common. For this, let $I_1, \dots, I_{nd}$ be an i.i.d. sequence of symmetric Bernoulli variables independent of $\mathbf{X}$:

$$\mathbb{P}(I_t = 0) = \mathbb{P}(I_t = 1) = 1/2, \quad t = 1, \dots, nd.$$

We define $Y_1, Y_2, \dots$ inductively. Fix $t \geqslant 0$. Suppose that we have already revealed the values $Y_1, \dots, Y_t$. If

$$2\frac{d - \deg_t(v)}{nd - t} - \frac{1}{n} \geqslant 0 \quad \text{for every } v \in [n], \tag{3.1}$$

then generate an auxiliary random variable Z_{t+1} independently of I_{t+1} according to the following distribution (note that the left-hand side of (3.1) sums over $v \in [n]$ to 1)

$$\mathbb{P}\big(Z_{t+1} = v \mid Y_1, \dots, Y_t\big) = 2\frac{d - \deg_t(v)}{nd - t} - \frac{1}{n}, \quad v = 1, \dots, n.$$

If (3.1) holds, set $Y_{t+1} = I_{t+1}X_{t+1} + (1 - I_{t+1})Z_{t+1}$. Otherwise generate Y_{t+1} directly according to the conditional probabilities (2.1). The distribution of Z_{t+1} is chosen precisely in such a way that (2.1) holds for any values of variables $Y_1, \dots, Y_t$, regardless of whether (3.1) is satisfied or not. This guarantees that $\mathbf{Y} = (Y_1, \dots, Y_{nd})$ is actually uniformly distributed over $\mathcal{S}$.

The following lemma states that we can embed $\mathbb{H}^{(k)}(n, m)$ in the red subgraph of $\mathbb{H}_*^{(k)}(n, d)$.

Lemma 3.1. *For every $k \geqslant 3$, there is a constant $C > 0$ such that if $d \geqslant C \log n$ and $d = o(n^{1/2})$, then one can define a joint distribution of $\mathbb{H}^{(k)}(n, m)$ and $\mathbf{Y}$ in such a way that*

$$\mathbb{H}^{(k)}(n, m) \subset \mathsf{H}(\mathbf{Y}_{\mathrm{red}}) \quad \textit{a.a.s.}$$

Proof. Let

$$W = \{0 \leqslant i \leqslant rm - 1 : I_{ki+1} = \cdots = I_{ki+k} = 1\}$$

and let $\mathbf{X}'$ be the subsequence of $\mathbf{X}$ formed by concatenation of the edges $(X_{ki+1}, \dots, X_{ki+k})$, $i \in W$. Define the events

$$\mathcal{A} = \{\mathbf{X} \text{ has no multiple edges}, \lambda(\mathbf{X}) \leqslant L, |W| \geqslant m + L\},$$
$$\mathcal{B} = \{\text{inequality (3.1) holds for every } v \in [n] \text{ and } t < krm\}.$$

Suppose that $\mathcal{A}$ holds. Then all edges of $\mathbf{X}'$ are distinct and at least m of them are proper. By symmetry, we can take, say, the first m of these edges to form $\mathbb{H}^{(k)}(n, m)$. If $\mathcal{A}$ fails, we simply generate $\mathbb{H}^{(k)}(n, m)$ independently of everything else.

Further, if $\mathcal{B}$ holds, then for every $i \in W$ we have

$$(Y_{ki+1}, \dots, Y_{ki+k}) = (X_{ki+1}, \dots, X_{ki+k}),$$

which is to say that $\mathsf{H}(\mathbf{X}')$ is a subgraph of $\mathsf{H}(\mathbf{Y}_{\mathrm{red}})$. Consequently,

$$\mathbb{P}(\mathbb{H}^{(k)}(n, m) \subset \mathsf{H}(\mathbf{Y}_{\mathrm{red}})) \geqslant \mathbb{P}(\mathcal{A} \cap \mathcal{B}),$$

so it is enough to show that each of the events $\mathcal{A}$ and $\mathcal{B}$ holds a.a.s.

By Proposition 2.1, the first two conditions defining $\mathcal{A}$ hold a.a.s. As for the last one, note that $|W| \sim \mathrm{Bi}(rm, 2^{-k})$, therefore $\mathbb{E}|W| = (1 + 2^{-k})m$ and $\mathrm{Var}\,|W| = O(m)$. Since $L = o(m)$, Chebyshev's inequality implies that, for n large enough,

$$\mathbb{P}(|W| < m + L) \leqslant \frac{\mathrm{Var}\,|W|}{(2^{-k}m - L)^2} = O(m^{-1}) = o(1).$$

Concerning the event $\mathcal{B}$, if for some $t < krm$ and some $v \in [n]$ inequality (3.1) does not hold, then $\deg_t(v) > d/2$, and consequently $\deg_{krm}(v) > d/2$. Note that $\deg_{krm}(v)$, $v = 1, \dots, n$, are identically distributed hypergeometric random variables. Let $X := \deg_{krm}(1)$. The probability that $\mathcal{B}$ fails is thus at most

$$\mathbb{P}(\deg_{krm}(v) > d/2 \text{ for some } v \in [n]) \leqslant n\mathbb{P}(X > d/2).$$

We have $\mathbb{E}X = krm/n \leqslant rcd$. Since $c < 1/2r$, applying, say, Theorem 2.10 from [13], we obtain

$$\mathbb{P}(X > d/2) \leqslant \exp\{-ad\} \leqslant \exp\{-aC \log n\},$$

for some positive constant a. Choosing $C > a^{-1}$ we get $n\mathbb{P}(X > d/2) = o(1)$, thus concluding the proof. □

4. Getting rid of red loops

Let $\mathcal{E}$ be the family of sequences in $\mathcal{S}$ with no multiple edges and containing at most L loops, but no loops of type other than $x_1x_1x_2\dots x_{k-1}$ (up to reordering of vertices), where $x_1,\dots,x_{k-1}$ are distinct. By Proposition 2.2 we have that $\mathbf{Y} \in \mathcal{E}$ a.a.s. Partition $\mathcal{E}$ according to the number of loops into sets

$$\mathcal{E}_l := \{\mathbf{y} \in \mathcal{E} : \lambda(\mathbf{y}) = l\}, \quad l = 0,\dots,L.$$

Let $\mathcal{G}_l$ be the family of those sequences in $\mathcal{E}_l$ which contain no red loops. Note that $\mathcal{G}_0 = \mathcal{E}_0$ consists precisely of those sequences $\mathbf{y} \in \mathcal{S}$ for which $\mathsf{H}(\mathbf{y})$ is simple.

Condition on $\mathbf{Y} \in \mathcal{E}$ and let $\mathbf{Y}'$ be a sequence obtained from $\mathbf{Y}$ by swapping the red loops of $\mathbf{Y}$ (if any) with a subset of green proper edges chosen uniformly at random. More formally, let $f_1,\dots,f_r$ be the red loops and let $e_1,\dots,e_g$ be the green proper edges of $\mathbf{Y}$ in the order they appear in $\mathbf{Y}$. Pick a set of indices $1 \leqslant i_1 < \cdots < i_r \leqslant g$ uniformly at random, and swap f_j with e_{i_j} for $j = 1,\dots,r$, preserving the order of vertices inside the edges. Note that this does not change the underlying k-multigraph, that is, $\mathsf{H}(\mathbf{Y}) = \mathsf{H}(\mathbf{Y}')$.

Proposition 4.1. *$\mathbf{Y}'$ is uniform on each $\mathcal{G}_l$, $l = 0,\dots,L$.*

Proof. Fix l. Clearly $\mathbf{Y}' \in \mathcal{G}_l$ if and only if $\mathbf{Y} \in \mathcal{E}_l$. Also, $\mathbf{Y}$ is uniform on $\mathcal{E}_l$. For integer $r \in [0,l]$, every $\mathbf{z} \in \mathcal{G}_l$ can be obtained from the same number (say, b_r) of $\mathbf{y}$ in $\mathcal{E}_l$ with exactly r red loops. On the other hand, for every $\mathbf{y}$ with exactly r red loops there is the same number (say, a_r) of $\mathbf{z}$ in $\mathcal{G}_l$ that can be obtained from $\mathbf{y}$. Hence, for every $\mathbf{z} \in \mathcal{G}_l$,

$$\mathbb{P}\big(\mathbf{Y}' = \mathbf{z} \,|\, \mathbf{Y} \in \mathcal{E}_l\big) = \sum_{r=0}^{l} \frac{b_r}{a_r|\mathcal{E}_l|},$$

which is the same for all $\mathbf{z} \in \mathcal{G}_l$. □

The following technical result will be used in the next section. Let

$$\tilde{\mathcal{S}} := \{\mathbf{y} \in \mathcal{S} : |\varphi(\mathbf{y}) - \mathbb{E}\varphi(\mathbf{Y})| \leqslant n^{3/4}d\}.$$

Proposition 4.2. *If $d = o(n^{1/2})$, then*

$$\mathbb{P}(\mathbf{Y}' \in \tilde{\mathcal{S}}) = 1 - o(1).$$

Proof. Suppose $\mathbf{z}$ is obtained from $\mathbf{y}$ by swapping a red loop with a green proper edge. This affects the green degree of at most $2k-1$ vertices v, and for every such v we have

$$|(\deg_{\text{green}}(\mathbf{y}; v))_2 - (\deg_{\text{green}}(\mathbf{z}; v))_2| = O(d),$$

uniformly for all such $\mathbf{y}, \mathbf{z}$. Hence, uniformly

$$|\varphi(\mathbf{Y}) - \varphi(\mathbf{Y}')| = O(Ld), \quad \mathbf{Y} \in \mathcal{E}.$$

By Proposition 2.2 we have that $\mathbf{Y} \in \mathcal{E}$ a.a.s. Hence,

$$\begin{aligned}\mathbb{P}(\mathbf{Y}' \notin \tilde{\mathcal{S}}) &\leqslant \mathbb{P}\big(|\varphi(\mathbf{Y}) - \mathbb{E}\varphi(\mathbf{Y})| > n^{3/4}d - O(Ld) \mid \mathbf{Y} \in \mathcal{E}\big) \\ &\sim \mathbb{P}\big(|\varphi(\mathbf{Y}) - \mathbb{E}\varphi(\mathbf{Y})| > n^{3/4}d - O(Ld)\big).\end{aligned}$$

Finally, since $d = o(n^{1/2})$, the last probability tends to zero by (2.4). □

5. Getting rid of green loops

In this section we complete the proof of Theorem 1.1, deferring the proofs of two technical results to the next section. By Lemma 3.1, which we proved in Section 3, the random k-multigraph $\mathsf{H}(\mathbf{Y}_{\text{red}})$ contains $\mathbb{H}^{(k)}(n,m)$ a.a.s. As $\mathbb{H}^{(k)}(n,m) \subset \mathsf{H}(\mathbf{Y})$ implies that $\mathbb{H}^{(k)}(n,m) \subset \mathsf{H}(\mathbf{Y}')$, it remains to define a procedure which a.a.s. transforms $\mathbf{Y}'$ (leaving the red edges of $\mathbf{Y}'$ intact) into a random k-graph distributed approximately as $\mathbb{H}^{(k)}(n,d)$.

For this we define an operation which decreases the number of green loops one at a time. Two sequences $\mathbf{y} \in \mathcal{G}_l$, $\mathbf{z} \in \mathcal{G}_{l-1}$ are said to be *switchable* if $\mathbf{z}$ can be obtained from $\mathbf{y}$ by the following operation, called a *switching*, which is a generalization (to $k \geqslant 3$) of a switching defined by McKay and Wormald [18] for $k = 2$. Among the edges of $\mathbf{y}$, choose a loop f and an ordered pair (e_1, e_2) of green proper edges: see Figure 1(a). Putting $s = |e_1 \cap e_2|$ and ignoring the order of the vertices inside the edges, one can write

$$f = vvx_1 \dots x_{k-2}, \quad e_1 = w_1 \dots w_s y_1 \dots y_{k-s}, \quad e_2 = w_1 \dots w_s z_1 \dots z_{k-s}.$$

Loop f contains two copies of v, the left one and the right one (with respect to their order in the sequence $\mathbf{y}$). Select vertices $y_* \in \{y_1, \dots, y_{k-s}\}$ and $z_* \in \{z_1, \dots, z_{k-s}\}$, and swap y_* with the left copy of v and z_* with the right one. The effect of switching is that f, e_1, and e_2 are replaced by three proper edges (see Figure 1(b)):

$$e_1' = e_1 \cup \{v\} - \{y_*\}, \quad e_2' = e_2 \cup \{v\} - \{z_*\}, \quad e_3' = f \cup \{y_*, z_*\} - \{v, v\}.$$

A *backward switching* is the reverse operation that reconstructs $\mathbf{y} \in \mathcal{G}_l$ from $\mathbf{z} \in \mathcal{G}_{l-1}$. It is performed by choosing a vertex v, an ordered pair of green proper edges e_1', e_2' containing v, one more green proper edge e_3', choosing a pair of vertices $y, z \in e_3'$, and swapping y with the copy of v in e_1 and z with the one in e_2.

Note that, given f, e_1, e_2, not every choice of y_*, z_* defines a forward switching, due to possible creation of new loops or multiple edges. We say that the choices of y_*, z_* which

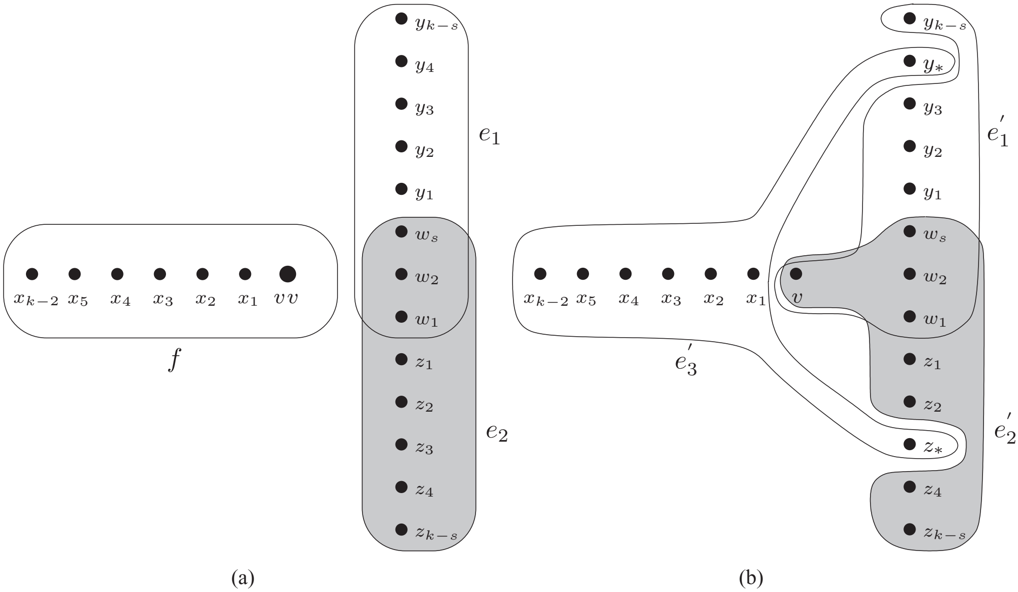


Figure 1. Edges affected by a switching (a) before and (b) after.

do define a switching are *admissible*. Similarly a choice of y, z is *admissible* with respect to v, e_1', e_2', and e_3' if it defines a backward switching.

Given $\mathbf{y} \in \mathcal{G}_l$, let $F(\mathbf{y})$ and $B(\mathbf{y})$ be the number of ways to perform forward switching and backward switching, respectively.

Let Sw denote a (random) operation which, given $\mathbf{y} \in \mathcal{G}_l$, applies to it a forward switching, chosen uniformly at random from the $F(\mathbf{y})$ possibilities. Let $\mathbf{Y}'' \in \mathcal{G}_0$ be the sequence obtained from $\mathbf{Y}'$ by applying Sw until there are no loops left, namely, $\lambda(\mathbf{Y}')$ times. Suppose for a moment that for every l and $\mathbf{y} \in \mathcal{G}_l$ functions $F(\mathbf{y})$ and $B(\mathbf{y})$ depend on l, but not on the actual choice of $\mathbf{y}$. If this were true, then, as one could easily show, $\mathbf{Y}''$ would be uniform over $\mathcal{G}_0$. As we will see, we are not far from this idealized setting, because Proposition 5.1(a) below implies that $F(\mathbf{y})$ is essentially proportional to $l = \lambda(\mathbf{y})$. On the other hand, Proposition 5.1(b) shows that $B(\mathbf{y})$ depends on a more complicated parameter of $\mathbf{y}$, namely on $\varphi(\mathbf{y})$ defined in Section 2.

To make $B(\mathbf{y})$ essentially independent of $\mathbf{y}$, we will apply switchings not to every element of $\mathcal{G}_0 \cup \cdots \cup \mathcal{G}_L$, but to a slightly smaller subfamily. Let

$$\tilde{\mathcal{G}}_l := \mathcal{G}_l \cap \tilde{\mathcal{S}}, \quad l = 0, \ldots, L,$$

where $\tilde{\mathcal{S}}$ has been defined in the previous section.

We condition on $\mathbf{Y}' \in \tilde{\mathcal{S}}$ and deterministically map $\mathbf{Y}''$ to a simple k-graph

$$\tilde{\mathbb{H}}^{(k)}(n, d) := \mathsf{H}(\mathbf{Y}'').$$

Note that switching does not affect the green degrees, and thus does not change the value of φ. Therefore, if one applies a forward or backward switching to a sequence $\mathbf{y} \in \tilde{\mathcal{S}}$, the resulting sequence is also in $\tilde{\mathcal{S}}$. Moreover, Proposition 4.2 shows that by restricting $\mathbf{Y}'$ to $\tilde{\mathcal{S}}$, we do not exclude many sequences.

The following proposition quantifies the amount by which a single application of Sw distorts the uniformity of $\mathbf{Y}'$.

Proposition 5.1. *If $1 \leqslant d = o(n^{1/2})$, then*

(a) *for $\mathbf{y} \in \mathcal{G}_l, 0 < l \leqslant L$,*

$$k^2 l(M-rm)^2\left(1-O\left(\frac{L+d^2}{M}\right)\right) \leqslant F(\mathbf{y}) \leqslant k^2 l(M-rm)^2;$$

(b) *for $\mathbf{y} \in \mathcal{G}_l$, $0 \leqslant 1 < L$,*

$$\binom{k}{2}(\varphi(\mathbf{y})-2kLd)(M-rm)\left(1-O\left(\frac{L+d^2}{M}\right)\right) \leqslant B(\mathbf{y}) \leqslant \binom{k}{2}\varphi(\mathbf{y})(M-rm);$$

(b′) *for $\mathbf{y} \in \tilde{\mathcal{G}}_l$, $0 \leqslant 1 < L$,*

$$\binom{k}{2}\mathbb{E}\varphi(\mathbf{Y})(M-rm)\left(1-O\left(\frac{n^{3/4}d}{\mathbb{E}\varphi(\mathbf{Y})}+\frac{L+d^2}{M}\right)\right)$$
$$\leqslant B(\mathbf{y}) \leqslant \binom{k}{2}\mathbb{E}\varphi(\mathbf{Y})(M-rm)\left(1+O\left(\frac{n^{3/4}d}{\mathbb{E}\varphi(\mathbf{Y})}\right)\right).$$

Finally, we need to show that the final step of the procedure, that is, the mapping of $\mathbf{Y}''$ to $\mathsf{H}(\mathbf{Y}'')$, has negligible influence on the uniformity of the distribution. For this, set

$$P_H := |\mathsf{H}^{-1}(H) \cap \tilde{\mathcal{G}}_0| = |\{\mathbf{y} \in \tilde{\mathcal{G}}_0 : \mathsf{H}(\mathbf{y}) = H\}|, \quad H \in \mathcal{H}^{(k)}(n,d).$$

Proposition 5.2. *If $d = o(n^{1/2})$, then uniformly for every $H \in \mathcal{H}^{(k)}(n,d)$*

$$(1-o(1))M!(k!)^M \leqslant P_H \leqslant M!(k!)^M.$$

Proofs of Propositions 5.1 and 5.2 can be found in Section 6.

Lemma 5.3. *There is a sequence $\varepsilon_n = o(1)$ such that, for every $H \in \mathcal{H}^{(k)}(n,d)$,*

$$\mathbb{P}(\tilde{\mathbb{H}}^{(k)}(n,d) = H) = (1 \pm \varepsilon_n)|\mathcal{H}^{(k)}(n,d)|^{-1}.$$

Proof. Clearly it is enough to show that for some function $p = p(n,l)$ we have

$$\mathbb{P}\big(\tilde{\mathbb{H}}^{(k)}(n,d) = H \mid \mathbf{Y}' \in \tilde{\mathcal{G}}_l\big) = (1+o(1))p(n,l) \tag{5.1}$$

uniformly for $l \leqslant L$ and $H \in \mathcal{H}^{(k)}(n,d)$. Indeed,

$$\mathbb{P}(\tilde{\mathbb{H}}^{(k)}(n,d) = H) = \sum_{l=0}^{L}\mathbb{P}\big(\tilde{\mathbb{H}}^{(k)}(n,d) = H \mid \mathbf{Y}' \in \tilde{\mathcal{G}}_l\big)\mathbb{P}(\mathbf{Y}' \in \tilde{\mathcal{G}}_l) = (1+o(1))p(n),$$

where $p(n) := \sum_l p(n,l)\mathbb{P}(\mathbf{Y}' \in \tilde{\mathcal{G}}_l)$ is independent of H.

Let $F_l = k^2 l(M - rm)^2$ and $B = \binom{k}{2}\,\mathbb{E}\varphi(\mathbf{Y})(M - rm)$ be the asymptotic values of the bounds in Proposition 5.1, (a) and (b′), respectively.

By Proposition 4.1, we can treat $\mathbf{Y}'$ as a uniformly chosen element of $\tilde{\mathcal{G}}_l = \mathcal{G}_l \cap \tilde{\mathcal{S}}$. Every realization of l switchings that generate $\mathbf{Y}''$ produces a *trajectory*

$$(\mathbf{y}^{(l)}, \ldots, \mathbf{y}^{(0)}) \in \tilde{\mathcal{G}}_l \times \cdots \times \tilde{\mathcal{G}}_0,$$

where $\mathbf{y}^{(k)}$ is switchable with $\mathbf{y}^{(k-1)}$ for $k = 1, \ldots, l$. The probability that a particular such trajectory occurs is

$$\begin{aligned}\frac{1}{|\tilde{\mathcal{G}}_l| F(\mathbf{y}^{(l)}) \ldots F(\mathbf{y}^{(1)})} &= \left(1 + O\left(\frac{L + d^2}{M}\right)\right)^l |\tilde{\mathcal{G}}_l|^{-1} \prod_{i=1}^{l} F_i^{-1} \\ &= (1 + o(1))|\tilde{\mathcal{G}}_l|^{-1} \prod_{i=1}^{l} F_i^{-1},\end{aligned} \tag{5.2}$$

the first equation following from Proposition 5.1.

On the other hand, by Propositions 5.1 and 5.2, the number of trajectories that lead to a particular $H \in \mathcal{H}^{(k)}(n,d)$ is

$$P_H B^l \left(1 + O\left(\frac{n^{3/4} d}{\mathbb{E}\varphi(\mathbf{Y})} + \frac{L + d^2}{M}\right)\right)^l = (1 + o(1))M!(k!)^M B^l, \tag{5.3}$$

because $\mathbb{E}\varphi(\mathbf{Y}) = \Theta(nd^2)$ by (2.3). Now the estimate (5.1) with

$$p(n,l) = M!(k!)^M B^l |\tilde{\mathcal{G}}_l|^{-1} \prod_{i=1}^{l} F_i^{-1}$$

follows by multiplication of (5.2) and (5.3). □

Proof of Theorem 1.1. Let μ be a uniform distribution over $\mathcal{H}^{(k)}(n,d)$ and let v be the distribution of $\tilde{\mathbb{H}}^{(k)}(n,d)$, that is,

$$\mu(H) = |\mathcal{H}^{(k)}(n,d)|^{-1}, \quad v(H) = \mathbb{P}(\tilde{\mathbb{H}}^{(k)}(n,d) = H), \quad H \in \mathcal{H}^{(k)}(n,d).$$

By Lemma 5.3 the *total variation distance* between the measures μ and v is

$$\mathrm{d}_{TV}(\mu, v) := \frac{1}{2} \sum_{H \in \mathcal{H}^{(k)}(n,d)} |\mu(H) - v(H)| \leqslant \frac{1}{2} \sum_{H} \varepsilon_n \mu(H) = o(1).$$

Therefore a standard fact from probability theory (see, *e.g.*, [1, p. 254]) implies that there is a joint distribution of $\tilde{\mathbb{H}}^{(k)}(n,d)$ and $\mathbb{H}^{(k)}(n,d)$ such that

$$\tilde{\mathbb{H}}^{(k)}(n,d) = \mathbb{H}^{(k)}(n,d) \quad \text{a.a.s.} \tag{5.4}$$

By definition of $\tilde{\mathbb{H}}^{(k)}(n,d)$, if $\mathbb{H}^{(k)}(n,m) \subset \mathsf{H}(\mathbf{Y}_{\mathrm{red}})$, then $\mathbb{H}^{(k)}(n,m) \subset \tilde{\mathbb{H}}^{(k)}(n,d)$. Therefore, Theorem 1.1 follows by Lemma 3.1 and Proposition 4.2. □

6. Remaining proofs

Proof of Proposition 5.1. **(a)** The upper bound follows from the fact that after we choose (in one of at most $l(M-rm)^2$ ways) a loop and two green edges, we have at most k^2 admissible choices of vertices y_* and z_*.

We say that two edges e', e'' of a k-graph are *distant* from each other if they do not intersect and there is no third edge e''' that intersects both e' and e''. Note that for any edge e there are at most $k^2 d^2$ edges *not* distant from e.

For the lower bound, let us estimate the number of triples (f, e_1, e_2) for which we have exactly k^2 admissible choices of y_*, z_*. For this it is sufficient that $e_1 \cap e_2 = \emptyset$ and both e_1 and e_2 are distant from f in $\mathsf{H}(\mathbf{y})$. Given f, we can choose such e_1 in at least

$$M - rm - l - k^2 d^2 = (M - rm)(1 - O((L + d^2)/M))$$

ways and then choose such e_2 in at least

$$M - rm - l - k^2 d^2 - kd = (M - rm)(1 - O((L + d^2)/M))$$

ways. Hence the lower bound.

(b) We can choose a vertex $v \in [n]$ and an ordered pair of edges e_1', e_2' containing v in at most $\varphi(\mathbf{y})$ ways and then choose e_3' in at most $M - rm$ ways. The number of admissible choices of vertices $y, z \in e_3'$ is at most $\binom{k}{2}$, which gives the upper bound.

For the lower bound, we estimate the number of quadruples v, e_1', e_2', e_3' for which there are exactly $\binom{k}{2}$ admissible choices of y, z. For this it is sufficient that e_3' is distant from both e_1' and e_2' in $\mathsf{H}(\mathbf{y})$. The number of ways to choose v, e_1', e_2' is exactly

$$\sum_{v \in [n]} \left(\deg'_{\text{green}}(\mathbf{y}; v)\right)_2, \tag{6.1}$$

where $\deg'_{\text{green}}(\mathbf{y}; v)$ is the number of green *proper* edges containing vertex v. It is obvious that (6.1) is at most $\varphi(\mathbf{y})$ and, as one can easily see, at least $\varphi(\mathbf{y}) - 2kLd$. The lower bound now follows, since, given v, e_1', e_2', we can choose e_3' in at least

$$M - rm - l - 4k^2 d^2 = (M - rm)(1 - O((L + d^2)/M)$$

ways.

(b′) This is immediate from (b) and the definition of $\tilde{\mathcal{G}}_l$. $\square$

Proof of Proposition 5.2. The upper bound is just $|\mathsf{H}^{-1}(H)|$. For the lower bound, we let $\mathbf{Y}|_H$ be a sequence chosen uniformly at random from $\mathsf{H}^{-1}(H)$ and show that the probabilities

$$\mathbb{P}(|\varphi(\mathbf{Y}|_H) - \mathbb{E}\varphi(\mathbf{Y})| > n^{3/4} d), \quad H \in \mathcal{H}^{(k)}(n, d),$$

uniformly tend to zero. Since φ does not depend on the order of vertices inside the edges of $\mathbf{Y}$, we can treat $\mathbf{Y}|_H$ as a random permutation of the M edges of H, which we denote by $e_1, \ldots, e_M$. Since H is simple, we have

$$\varphi(\mathbf{Y}|_H) = \sum_{v \in [n]} \sum_{\substack{e_i, e_j \ni v \\ i \neq j}} \mathbb{I}_{\{e_i, e_j \text{ are green in } \mathbf{Y}|_H\}},$$

whence

$$\mathbb{E}\varphi(\mathbf{Y}|_H) = n(d)_2\frac{(M-rm)_2}{(M)_2}.$$

Therefore (2.3) and simple calculations yield

$$\mathbb{E}\varphi(\mathbf{Y}|_H) - \mathbb{E}\varphi(\mathbf{Y}) = O(nd^2M^{-1}) = O(d).$$

Further, if $\mathbf{y}, \mathbf{z} \in \mathsf{H}^{-1}(H)$ and $\mathbf{z}$ can be obtained from $\mathbf{y}$ by swapping two edges, then

$$|\varphi(\mathbf{y}) - \varphi(\mathbf{z})| = O(d),$$

uniformly for all such $\mathbf{y}$ and $\mathbf{z}$. Therefore (2.2) applies to $f = \varphi$ with $N = M$ and $b = O(d)$. To sum up,

$$\begin{aligned}\mathbb{P}\big(|\varphi(\mathbf{Y}|_H) - \mathbb{E}\varphi(\mathbf{Y})| > n^{3/4}d\big) &\leqslant \mathbb{P}\big(|\varphi(\mathbf{Y}|_H) - \mathbb{E}\varphi(\mathbf{Y}|_H)| > n^{3/4}d - O(d)\big)\\ &\leqslant 2\exp\left\{-\frac{(n^{3/4}d - O(d))^2}{O(Md^2)}\right\} = o(1),\end{aligned}$$

the equation following from the assumption $d = o(n^{1/2})$. □

7. Concluding remarks

Remark 1. Theorem 1.1 is closely related to a result of Kim and Vu [14], who proved, for d growing faster than $\log n$ but more slowly than $n^{1/3}/\log^2 n$, that there is a joint distribution of $\mathbb{H}^{(2)}(n,p)$ and $\mathbb{H}^{(2)}(n,d)$ with p satisfying $p \sim d/n$ so that

$$\mathbb{H}^{(2)}(n,p) \subset \mathbb{H}^{(2)}(n,d) \quad \text{a.a.s.} \tag{7.1}$$

It is known (see, *e.g.*, [4]) that $\mathbb{H}^{(2)}(n,p)$ is a.a.s. Hamiltonian, when the expected degree $(n-1)p$ grows faster than $\log n$. Therefore (7.1) implies an analogue of Theorem 1.4 for graphs.

Remark 2. In [11] the authors used the same switching as in the present paper to count d-regular k-graphs approximately for $k \geqslant 3$ and $1 \leqslant d = o(n^{1/2})$ as well as for $k \geqslant 4$ and $d = o(n)$. The application of the technique is somewhat easier there, because there is no need to preserve the red edges. The restriction $d = o(n^{1/2})$ that appears in Theorem 1.1 also has a natural meaning in [11], since the counting formula there gives the asymptotics of the probability $p_{n,d} := \mathbb{P}(\mathbb{H}^{(k)}_*(n,d)$ is simple) for $d = o(n^{1/2})$, while for $k \geqslant 4$ and $n^{1/2} \leqslant d = o(n)$ it just gives the asymptotics of $\log p_{n,d}$.

Remark 3. The lower bound on d in Theorem 1.1 is necessary because the second moment method applied to $\mathbb{H}^{(k)}(n,p)$ (see [3, Theorem 3.1(ii)]) and asymptotic equivalence of $\mathbb{H}^{(k)}(n,p)$ and $\mathbb{H}^{(k)}(n,m)$ yields that for $d = o(\log n)$ and $m \sim cM$ there is a sequence $\Delta = \Delta(n)$ such that $d = o(\Delta)$ and the maximum degree $\mathbb{H}^{(k)}(n,m)$ is at least Δ a.a.s.

Remark 4. For d greater than $\log n$, however, the degree sequence of $\mathbb{H}^{(k)}(n,p)$ is closely concentrated around the expected degree. Therefore it is plausible that Theorem 1.1 can be extended to d greater than $n^{1/2}$. However, $n^{1/2}$ seems to be an obstacle which cannot be overcome without a proper refinement of our proof.

Remark 5. In view of Remark 3, our approach cannot be extended to $d = O(\log n)$. Nevertheless, we believe that the following extension of Theorem 1.4 is valid.

Conjecture 1. *For every $k \geqslant 3$ there is a constant $d_0 = d_0(k)$ such that, for any $d \geqslant d_0$,*

$$\mathbb{H}^{(k)}(n,d) \text{ contains a loose Hamilton cycle a.a.s.}$$

Recall that Robinson and Wormald [19, 20] proved for $k = 2$ that as far as fixed d is concerned, it suffices to take $d \geqslant 3$. Their approach is based on a very careful analysis of variance of a random variable counting the number of Hamilton cycles in the configuration model. Unfortunately, for $k \geqslant 3$ similar computations become extremely complicated and involved, discouraging one from taking this approach.

Remark 6. In this paper, we were concerned only with loose cycles. One can also consider a more general problem. Define an *ℓ-overlapping cycle* as a k-uniform hypergraph in which, for some cyclic ordering of its vertices, every edge consists of k consecutive vertices, and every two consecutive edges (in the natural ordering of the edges induced by the ordering of the vertices) share exactly ℓ vertices. (Clearly, $\ell = 1$ corresponds to loose cycles.) The thresholds for the existence of ℓ-overlapping Hamilton cycles in $\mathbb{H}^{(k)}(n,p)$ have been recently obtained in [9]. However, proving similar results for $\mathbb{H}^{(k)}(n,d)$ and arbitrarily $\ell \geqslant 2$ seem to be hard. Based on results from [9] we believe that the following is true.

Conjecture 2. *For every $k > \ell \geqslant 2$ if $d \gg n^{\ell-1}$, then*

$$\mathbb{H}^{(k)}(n,d) \text{ contains an } \ell\text{-overlapping Hamilton cycle a.a.s.}$$

References

[1] Barbour, A. D., Holst, L. and Janson, S. (1992) *Poisson Approximation*, Vol. 2 of *Oxford Studies in Probability*, Oxford Science Publications, The Clarendon Press.

[2] Bender, E. A. and Canfield, E. R. (1978) The asymptotic number of labeled graphs with given degree sequences. *J. Combin. Theory Ser. A* **24** 296–307.

[3] Bollobás, B. (1980) A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *Europ. J. Combin.* **1** 311–316.

[4] Bollobás, B. (2001) *Random Graphs*, second edition, Vol. 73 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press.

[5] Bollobás, B. and Frieze, A. M. (1985) On matchings and Hamiltonian cycles in random graphs. In *Random Graphs '83: Poznań 1983*, Vol. 118 of *North-Holland Mathematics Studies*, North-Holland, pp. 23–46.

[6] Chvátal, V. (1991) Almost all graphs with 1.44n edges are 3-colorable. *Random Struct. Alg.* **2** 11–28.

[7] Cooper, C., Frieze, A. and Reed, B. (2002) Random regular graphs of non-constant degree: Connectivity and Hamiltonicity. *Combin. Probab. Comput.* **11** 249–261.

[8] Dudek, A. and Frieze, A. (2011) Loose Hamilton cycles in random uniform hypergraphs. *Electron. J. Combin.* **18** #48.
[9] Dudek, A. and Frieze, A. (2013) Tight Hamilton cycles in random uniform hypergraphs. *Random Struct. Alg.* **42** 374–385.
[10] Dudek, A., Frieze, A., Loh, P.-S. and Speiss, S. (2012) Optimal divisibility conditions for loose Hamilton cycles in random hypergraphs. *Electron. J. Combin.* **19** #44.
[11] Dudek, A., Frieze, A., Ruciński, A. and Šileikis, M. (2013) Approximate counting of regular hypergraphs. *Inform. Process. Lett.* **113** 785–788.
[12] Frieze, A. (2010) Loose Hamilton cycles in random 3-uniform hypergraphs. *Electron. J. Combin.* **17** note 28.
[13] Janson, S., Łuczak, T. and Rucinski, A. (2000) *Random Graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience.
[14] Kim, J. H. and Vu, V. H. (2004) Sandwiching random graphs: Universality between random graph models. *Adv. Math.* **188** 444–469.
[15] Knuth, D. E. (1976) Big omicron and big omega and big theta. *SIGACT News* **8** 18–24.
[16] Krivelevich, M., Sudakov, B., Vu, V. H. and Wormald, N. C. (2001) Random regular graphs of high degree. *Random Struct. Alg.* **18** 346–363.
[17] McDiarmid, C. (1998) Concentration. In *Probabilistic Methods for Algorithmic Discrete Mathematics*, Vol. 16 of *Algorithms and Combinatorics*, Springer, pp. 195–248.
[18] McKay, B. D. and Wormald, N. C. (1990) Uniform generation of random regular graphs of moderate degree. *J. Algorithms* **11** 52–67.
[19] Robinson, R. W. and Wormald, N. C. (1992) Almost all cubic graphs are Hamiltonian. *Random Struct. Alg.* **3** 117–125.
[20] Robinson, R. W. and Wormald, N. C. (1994) Almost all regular graphs are Hamiltonian. *Random Struct. Alg.* **5** 363–374.
[21] Wormald, N. C. (1999) Models of random regular graphs. In *Surveys in Combinatorics: Canterbury 1999*, Vol. 267 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, pp. 239–298.