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UNIVERSALITY OF RANDOM GRAPHS*

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Abstract. We prove that asymptotically (as $n \to \infty$) almost all graphs with n vertices and $C_d\, n^{2-\frac{1}{2d}} \log^{\frac{1}{d}} n$ edges are universal with respect to the family of all graphs with maximum degree bounded by d. Moreover, we provide an efficient deterministic embedding algorithm for finding copies of bounded degree graphs in graphs satisfying certain pseudorandom properties. We also prove a counterpart result for random bipartite graphs, where the threshold number of edges is even smaller but the embedding is randomized.

Key words. random graphs, universality, bounded degree graphs, graph embedding

AMS subject classifications. 05C80, 05C60

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1. Introduction.

Given graphs H and G, an *embedding* of H into G is an injective edge-preserving map $f\colon V(H) \to V(G)$, that is, for every $e = \{u, v\} \in E(H)$, we have $f(e) = \{f(u), f(v)\} \in E(G)$. We shall say that a graph H *is contained in G as a subgraph* if there is an embedding of H into G. Given a family of graphs $\mathcal{H}$, we say that G is *universal with respect to $\mathcal{H}$*, or *$\mathcal{H}$-universal*, if every $H \in \mathcal{H}$ is contained in G as a subgraph.

Consider the probability space of all graphs on n labeled vertices in which every pair of vertices forms an edge, randomly and independently, with probability p. We use the notation $G_{n,p}$ to denote a graph chosen randomly according to this probability measure; i.e., for any graph G on n labeled vertices and with m edges, $\mathbb{P}[G_{n,p} = G] = p^m(1-p)^{\binom{n}{2}-m}$. We say that $G_{n,p}$ possesses a property Q *asymptotically almost surely* (a.a.s.) if $\mathbb{P}[G_{n,p} \in Q] = 1 - o(1)$.

The construction of sparse universal graphs for various families of graphs has received a considerable amount of attention; see, e.g., [1, 2, 3, 6, 8, 10, 13, 18] and their references. One is particularly interested in (almost) tight $\mathcal{H}$-universal graphs, i.e., graphs whose number of vertices is equal (or close) to $\max_{H\in\mathcal{H}} |V(H)|$.

In [6] it is proved that for all $\varepsilon > 0$ and $d > 0$ there exists $c > 0$ such that a.a.s. $G_{n,p}$, $p = c/n$, is $\mathcal{T} = \mathcal{T}\big(d, (1-\varepsilon)n\big)$-universal, where $\mathcal{T}$ is the family of trees with $(1-\varepsilon)n$ vertices and maximum degree at most d. (See [7] for a recent improvement of this result.) In a related paper [11], the authors obtained an algorithm for finding

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bounded degree trees in subgraphs of (n, d, λ)-graphs; in particular, the result of [6] is turned into an embedding algorithm. In this paper we study the universality of random graphs with respect to the family of all bounded degree graphs.

Let $d \in \mathbb{N}$ be a fixed constant, let $\mathcal{H}(n, d) = \{H \subset K_n : \Delta(H) \leq d\}$ denote the class of (pairwise nonisomorphic) n-vertex graphs with maximum degree bounded by d, and let $\mathcal{H}(n, n; d) = \{H \subset K_{n,n} : \Delta(H) \leq d\}$ be the corresponding class for balanced bipartite graphs.

By counting all unlabeled d-regular graphs on n vertices one can easily show that every $\mathcal{H}(n, d)$-universal graph must have

$$M = \Omega(n^{2-2/d}) \tag{1}$$

edges (see [1] for details). This lower bound was almost matched by a construction from [2], which was subsequently improved in [3] and [4]. Those constructions were quite special and do not resemble a typical, or random, graph with the same number of edges. For that reason, in [1], the universality of random graphs was also studied.

For random graphs, slightly better lower bounds than (1) are known. Owing[1] to the threshold for the property that every vertex should belong to a copy of K_{d+1} (see [15, Theorem 3.22(i)]), the expected number of edges guaranteeing $\mathcal{H}(n, d)$-universality of $G_{n,p}$ must be at least $n^{2-2/(d+1)}(\log n)^{1/\binom{d+1}{2}}$, and similarly, by [15, Theorem 4.9], it must be at least $n^{2-2/(d+1)}$ for $\mathcal{H}(n, d)$-universality of $G_{(1+\varepsilon)n,p}$. Similar bounds apply to the random bipartite graph $G_{n,n,p}$.

In [1], it was proved that $G_{n,n,p}$ is a.a.s. $\mathcal{H}(n, n, d)$-universal if $p = cn^{-\frac{1}{2d}} \log^{\frac{1}{2d}} n$ and c is large enough and that $G_{(1+\varepsilon)n,p}$ is a.a.s. $\mathcal{H}(n, d)$-universal if $p = cn^{-\frac{1}{d}} \log^{\frac{1}{d}} n$ if c is large enough.

In this paper we prove two related results. The first one significantly pushes down the edge density p guaranteeing the universality of $G_{n,n,p}$.

THEOREM 1. *For every $d \in \mathbb{N}$ there exists C such that if $p = p(n) \geq Cn^{-1/d} \log^{1/d} n$, then the random bipartite graph $G_{n,n,p}$ is a.a.s. $\mathcal{H}(n, n, d)$-universal.*

The second result, at the cost of increase in p, establishes a tight universality of $G_{n,p}$ (and not of $G_{(1+\varepsilon)n,p}$) and provides, as opposed to Theorem 1, a deterministic embedding.

THEOREM 2. *For every $d \in \mathbb{N}$ there exists C such that if $p = p(n) \geq C\, n^{-1/(2d)} \log^{1/d} n$, then the random graph $G = G_{n,p}$ is a.a.s. $\mathcal{H}(n, d)$-universal. Moreover, for any $H \in \mathcal{H}(n, d)$, the embedding $H \hookrightarrow G$ can be constructed in deterministic polynomial time.*

Remark 3. Using the asymptotic equivalence between the two standard models of random graphs [15, Corollary 1.16] one can deduce from Theorem 2 that almost all graphs on n vertices with at least $C_d n^{2-1/(2d)} \log^{1/d} n$ edges are $\mathcal{H}(n, d)$-universal.

It would be interesting to establish the actual thresholds for the $\mathcal{H}(n, n; d)$-universality of $G_{n,n,p}$ and the $\mathcal{H}(n, d)$-universality of $G_{n,p}$.

In this paper we restrict our attention to $d \geq 2$ since the case $d = 1$ reduces to the threshold of containing a matching of maximum size (namely, a matching of size $\lfloor n/2 \rfloor$). It is well known [15, Chapter 4.1] that this threshold is $p \sim \log n/n$.

The proof of Theorem 1 is based on ideas from [20] and [19]. The embedding scheme used to prove Theorem 2 is inspired by the algorithmic version of the *blow-up lemma* of Komlós, Sárközy, and Szemerédi [17]. In their setting, they essentially

[1]Alternatively, we could use the threshold for the appearance of a K_{d+1}-factor in a random graph, which was determined by Johansson, Kahn, and Vu [16].

provided an algorithm to embed bounded degree spanning (bipartite) graphs into super-regular, dense, bipartite graphs. In our setting, we deal with sparse random graphs.

In section 2 we establish several typical properties of random graphs which imply universality. The proofs of Theorems 1 and 2 are presented in sections 3 and 4, respectively.

2. Properties of random graphs. In this section we establish properties of random graphs which will then be shown to guarantee the universality property with respect to bounded degree subgraphs.

We begin with some definitions.

DEFINITION 4. *Given a graph G, a vertex $v \in V(G)$, and a subset $\emptyset \neq S \subset V(G)$, denote by $G(v)$ the neighborhood of v in G and by*

$$G^{\cap}(S) = \bigcap_{v \in S} G(v)$$

the joint neighborhood of S in G. Moreover, we let $G^{\cap}(\emptyset) = V(G)$.

LEMMA 5. *For all $d \in \mathbb{N}$, $d \geq 2$, and $\gamma, \nu > 0$, if $p \geq C\, n^{-1/d} \log^{1/d} n$, where $C^d \geq \frac{d+2}{\gamma\nu}$, then the random bipartite graph $G = G_{n,n;p}$ with classes U and W together with a fixed subset $W' \subseteq W$, where $|W'| \geq \gamma n$ a.a.s. satisfies the following properties:*

(i) *For every $A \subset U$ (or $A \subset W$) with $|A| \leq d$*

$$(1-\nu)p^{|A|}n \leq |G^{\cap}(A)| \leq (1+\nu)p^{|A|}n.$$

(ii) *For every $U' \subset U$ with $|U'| \geq n/2$ there are at most $\frac{20}{p}$ vertices $w \in W$ such that $|G(w) \cap U'| < \frac{p}{2}\,|U'|$.*

(iii) *For every disjoint family $\mathcal{F} \subset \binom{U}{d}$ and a subset $T \subset W'$, with $|\mathcal{F}| \leq (1-\nu)|W'|$, and $|T| = |W'| - |\mathcal{F}| \geq \nu|W'|$, there exists a vertex $w \in T$ and a set $A \in \mathcal{F}$ such that $A \subset G(w)$.*

Proof. The first two properties are obtained by standard applications of the Chernoff inequality. Indeed, in (i), $Z_A := |G^{\cap}(A)|$ has a binomial distribution with expectation $\mathbb{E}Z_A = np^{|A|} \geq C^d \log n$, and so

$$\sum_{a=1}^{d} \binom{n}{a} \times \mathbb{P}(|Z_A - \mathbb{E}Z_A| \geq \nu\mathbb{E}Z_A) = o(1)$$

for sufficiently large C. To prove (ii) suppose that for some U' there is a subset $S \subset W$ of $20/p$ vertices $w \in W$ with $|G(w) \cap U'| < \frac{p}{2}\,|U'|$. Then there are fewer than $10|U'|$ edges between S and U', while the expected number of such edges is $20|U'|$. Thus,

$$\mathbb{P}\left(|G(w) \cap U'| < \frac{p}{2}\,|U'|\right) \leq \exp\{-\tfrac{1}{8}20|U'|\} \leq \exp\{-\tfrac{5}{4}n\}.$$

There are no more than 2^n choices of U' and $n^{20/p}$ choices of S and so the probability of the event opposite to that stated in part (ii) is $o(1)$. We will now prove property (iii).

Let $s, t \geq 1$ be such that $t \geq \nu\,|W'|$ and $s + t = |W'|$. For some fixed disjoint family $\mathcal{F} \subset \binom{U}{d}$ and $T \subset W'$ with $|\mathcal{F}| = s$ and $|T| = t$, the probability that there are no pairs$(w, A) \in T \times \mathcal{F}$ such that $A \subset G(w)$ is

$$(1-p^d)^{st} \leq \exp\{-p^d st\}.$$

The probability that there is a disjoint family $\mathcal{F} \subset \binom{U}{d}$ and $T \subset W'$ failing (iii) is at most

$$
\begin{aligned}
[*] &:= \sum_{t=\nu\,|W'|}^{|W'|-1} \binom{|W'|}{t} \binom{|U|}{d}^{|W'|-t} \exp\{-p^d t(|W'|-t)\} \\
&\leq \sum_t \exp\{(|W'|-t)\log n + (|W'|-t)d\log n - p^d t(|W'|-t)\} \\
&\leq \sum_t \exp\big\{(|W'|-t)\big[(d+1)\log n - p^d t\big]\big\}.
\end{aligned}
\tag{2}
$$

Observe that

$$p^d t \geq p^d \nu\, |W'| \geq C^d \nu\gamma \log n \geq (d+2)\log n,$$

and consequently $[*] \leq \sum_{j=1}^{\nu|W'|} n^{-j} = O(n^{-1}) = o(1)$. ∎

LEMMA 6. *For all $d \in \mathbb{N}$, $d \geq 2$, and $\varepsilon > 0$ there exists $C > 0$ such that if $p \geq C\, n^{-1/(2d)} \log^{1/d} n$, then the random graph $G = G_{n,p}$ a.a.s. satisfies the following properties:*

(i) $\delta(G) \geq (1-\varepsilon)pn$.

(ii) *For every pair of sets $A, B \subset V(G)$ with $p\,|A||B| \geq 100\varepsilon^{-3} n$ there are at least $(1-\varepsilon)|B|$ vertices $v \in B$ such that*

$$(1-\varepsilon)p|A| \leq |G(v) \cap A| \leq (1+\varepsilon)p|A|.$$

(iii) *For every $k \leq d$, $T \subset V(G)$ with $|T| \geq \sqrt{n}$ and every disjoint family $\mathcal{X} \subset \binom{V(G)\setminus T}{k}$ with $|\mathcal{X}| \geq \sqrt{n}$, we have*

$$(1-\varepsilon)p^k|T|\,|\mathcal{X}| \leq \Big|\big\{(w, X) \in T \times \mathcal{X} \,:\, X \subset G(w)\big\}\Big| \leq (1+\varepsilon)p^k|T|\,|\mathcal{X}|. \tag{3}$$

Proof. The first two properties are obtained by standard applications of the Chernoff inequality. We will now prove property (iii).

Let $k \leq d$ be fixed. For a choice of set T and family $\mathcal{X}$, the number of pairs (w, X) being counted is a binomial variable with mean $p^k|T|\,|\mathcal{X}|$. By the Chernoff inequality, the probability this variable deviates by more than $\varepsilon p^k|T|\,|\mathcal{X}|$ from the mean is at most

$$\exp\{-cp^k|T|\,|\mathcal{X}|\} \leq \exp\{-cp^d|T|\,|\mathcal{X}|\}$$

for a constant $c = c(\varepsilon)$.

On the other hand, the number of possible choices for T and the family $\mathcal{X}$ with predetermined cardinalities $t = |T|$ and $r = |\mathcal{X}|$ $(t, r \geq \sqrt{n})$ is at most $(n^d)^r n^t \leq \exp\{d(r+t)\log n\}$.

Since $p^d tr \geq \max\{C^d r\log n, C^d t\log n\}$, a large enough $C = C(d, \varepsilon)$ implies that

$$cp^d tr \geq 2d(r+t)\log n. \tag{4}$$

Therefore, we have

$$
\begin{aligned}
&\sum_{k=1}^{d} \sum_{T\,:\,|T|\geq\sqrt{n}} \sum_{\mathcal{X}\,:\,|\mathcal{X}|\geq\sqrt{n}} \mathbb{P}[(3) \text{ fails for } T, \mathcal{X}] \\
&\quad\leq \sum_{k=1}^{d} \sum_{t\geq\sqrt{n}} \sum_{r\geq\sqrt{n}} \exp\{d(r+t)\log n - cp^k tr\} \\
&\quad\overset{(4)}{\leq} dn^2 \exp\{-2d\sqrt{n}\log n\} = o(1).
\end{aligned}
\tag{5}
$$

Property (iii) then follows by the union bound. ∎

3. Universality of bipartite graphs. In this section we prove a slight strengthening of Theorem 1. Given $d, d' \in \mathbb{N}$, and a bipartite complete graph $K_{n,n}$ with vertex classes X and Y, $|X| = |Y| = n$, let

$$\mathcal{H}(n,n,d',d) = \big\{H \subset K_{n,n} \;:\; \deg_H(x) \le d' \text{ for } x \in X \text{ and } \deg_H(y) \le d \text{ for } y \in Y\big\}.$$

THEOREM 7. *For all $d, d' \in \mathbb{N}$, $2 \le d \le d'$, there exists C such that if $p = p(n) \ge Cn^{-1/d}\log^{1/d} n$, then the random bipartite graph $G_{n,n;p}$ is a.a.s. $\mathcal{H}(n,n,d',d)$-universal.*

Let $\mathcal{H}(n,n,d',=d)$ be defined as $\mathcal{H}(n,n,d',d)$ but with the additional condition that all vertices $y \in Y$ have degree *exactly* d. Note that if n is sufficiently larger than d', then for every $H' \in \mathcal{H}(n,n,d',d)$ there is an $H \in \mathcal{H}(n,n,d'+1,=d)$ such that $H' \subseteq H$. Thus, it suffices to show that $G_{n,n;p}$ is a.a.s. $\mathcal{H}(n,n,d'+1,=d)$-universal.

Let the two vertex classes of $G_{n,n;p}$ be U and W, $|U| = |W| = n$. For technical reasons we will need a partition of W. A partition in which the cardinalities of any two parts differ by at most 1 is called an *equipartition*. Let $W = W_1 \cup W_2 \cup \cdots \cup W_D$ be a fixed equipartition of W with

$$D := dd' + 1.$$

Notice that for every $i = 1, \dots, D$, we have

$$|W_i| \ge \left\lfloor \frac{n}{D} \right\rfloor > \frac{n}{D} - 1 \ge \frac{n}{D+1},$$

where the last inequality holds for $n \ge D(D+1)$.

Let

$$\nu = \frac{1}{8}(48e)^{-d}, \quad \gamma = \frac{1}{D+1}, \tag{6}$$

and let $C > 0$ be such that

$$C^d \ge \frac{d+2}{\gamma\nu}.$$

In particular, for our choice of $p = Cn^{-1/d}\log^{1/d} n$, Lemma 5 applies to $G_{n,n;p}$. Further, let G be a bipartite graph that for each $i = 1, \dots, D$ satisfies properties (i)–(iii) from Lemma 5 with $W' = W_i$. We will show that G contains all $H \in \mathcal{H}(n,n,d'+1,=d)$ as subgraphs, and consequently that G is $\mathcal{H}(n,n,d',d)$-universal. Theorem 7 will follow, since $G_{n,n;p}$ a.a.s. satisfies properties (i)–(iii) from Lemma 5.

Let us fix $H \in \mathcal{H}(n,n,d'+1,=d)$. In order to avoid certain dependency issues later in the proof, it would be convenient to assume that the d-element sets $H(y)$, $y \in Y$, are pairwise disjoint. This is not true in general, but it is possible to partition the set Y into finitely many subsets, each satisfying the above demand. A family of pairwise disjoint sets will be called a *disjoint family*.

Consider the auxiliary graph

$$J = (Y, \{uv \;:\; u, v \in Y,\ \mathrm{dist}_H(u,v) = 2\})$$

and note that $\Delta(J) \le dd'$. Similarly as in [14] we apply the Hajnal–Szemerédi Theorem [14] to J, thus obtaining an equipartition of $V(J)$ with D parts:

$$Y = Y_1 \cup \cdots \cup Y_D,$$

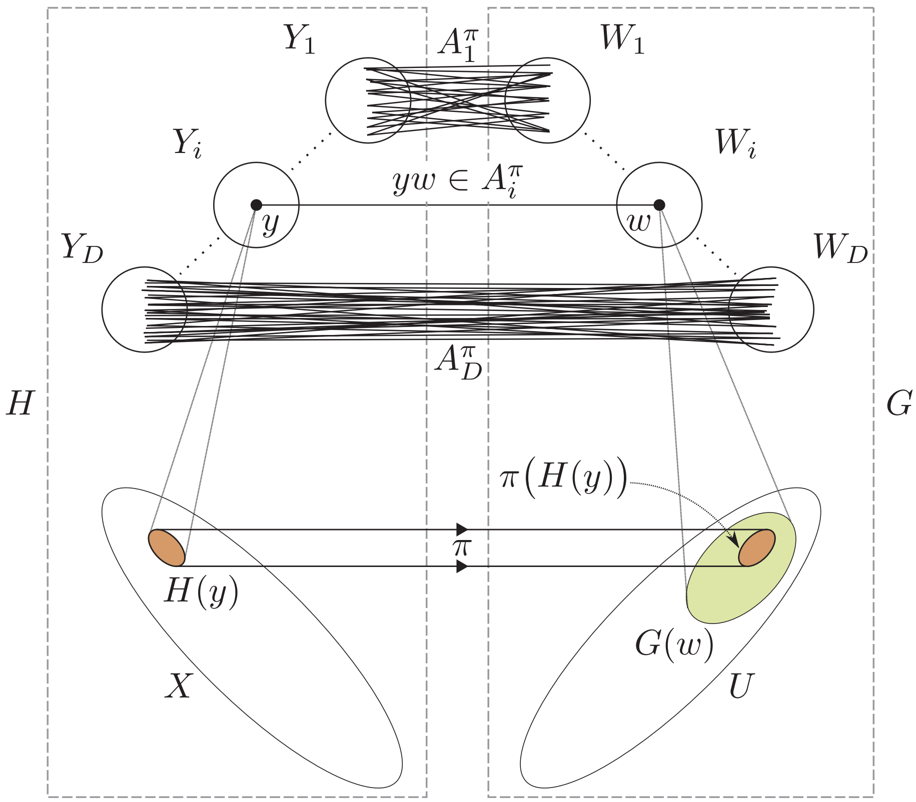

FIG. 1. *The idea of the proof of Theorem* 7.

where each Y_i is independent in J. Observe that by construction, for every $i = 1, \dots, D$, $\{H(y) \,:\, y \in Y_i\}$ is a disjoint family of d-element sets. We renumber the sets W_i so that $|Y_i| = |W_i|$ for all $i = 1, \dots, D$.

To show that $G \supseteq H$, our strategy is to find a bijection $\pi\colon X \to U$ which can be extended to an embedding f of H into G by selecting the images of vertices in Y. More precisely, given π, we will find a map $f\colon X \cup Y \to U \cup W$ such that

- $f|_X = \pi$,
- $f(Y_i) = W_i$ for all $i = 1, \dots, D$, and, most importantly,
- for all $y \in Y$

$$\pi(H(y)) \subseteq G(f(y)).$$

Let $\pi\colon X \to U$ be a random bijection and let A_i^π be the auxiliary bipartite graph with classes Y_i and W_i containing as edges all pairs $(y, w) \in Y_i \times W_i$ for which the π-image of the H-neighborhood of y is contained in the G-neighborhood of w (see Figure 1). Namely,

$$E(A_i^\pi) = \{(y, w) \in Y_i \times W_i \,:\, \pi(H(y)) \subseteq G(w)\}. \tag{7}$$

Suppose that each A_i^π, $i = 1, \dots, D$, contains a perfect matching M_i and set $M = \bigcup_{i=1}^D M_i$. Extend π to an embedding of H into G by letting, for every $y \in Y$, $f(y) = w$, where (y, w) is the edge of M incident to y. We claim that such an extension is an embedding of H into G.

The extension f is clearly a bijection. It remains to show that f is also edge-preserving. For an edge $e = (x, y) \in E(H)$, let i_e be such that $y \in Y_{i_e}$. By construction, $(y, f(y)) \in A^{\pi}_{i_e}$, which implies that $\pi(H(y)) \subseteq G(f(y))$ and thus $\pi(x) \in G(f(y))$. Consequently, $(f(x), f(y)) = (\pi(x), f(y)) \in E(G)$ and the map f is edge-preserving.

Therefore, in order to complete the proof of Theorem 7 it suffices to show the following probabilistic lemma. (Notice that the graph G in that lemma is fixed and the probability space in consideration refers to the random bijection π.)

LEMMA 8. *For every $i = 1, \ldots, D$, if G satisfies the properties listed in Lemma* 5 *with $W' = W_i$, then the graph A^{π}_i a.a.s. contains a perfect matching.*

Proof. Let us fix an index i throughout this proof and set

$$m := |Y_i| = |W_i|.$$

Recall that

$$\gamma n = \frac{n}{D+1} \leq m \leq \frac{n}{D-1} = \frac{n}{dd'}$$

for $n \geq D(D+1)$, where $D = dd' + 1$. We will verify Hall's condition in order to establish the result. To simplify notation, for every $V \subseteq V(A^{\pi}_i) = Y_i \cup W_i$ we set

$$N(V) = \bigcup_{v \in V} A^{\pi}_i(v). \tag{8}$$

It is well known that it suffices to show for some integer $m' \geq 0$ that

- $|N(S)| \geq |S|$ for all $S \subseteq Y_i$ with $|S| \leq m'$ and
- $|N(T)| \geq |T|$ for all $T \subseteq W_i$ with $|T| \leq m - m'$.

Set $m' = (1 - \nu)m$ and fix an *arbitrary* bijection $\pi\colon X \to U$. Observe that $\{\pi(H(y)) : y \in Y_i\}$ is a disjoint family. For all $S \subset Y_i$ and $T \subset W_i$ such that $|S| \leq m'$ and $|T| = m - |S|$, property (iii) from Lemma 5 yields that setting $\mathcal{F}_S = \{\pi(H(y)) : y \in S\} \subset \binom{U}{d}$, there is $(w, A) \in T \times \mathcal{F}_S$ satisfying $A \subseteq G(w)$. In particular, it follows that for *every* π we have $|N(S)| \geq |S|$ for all sets $S \subset Y_i$ with $|S| \leq m'$.

In remains to verify that Hall's condition holds a.a.s. for all sets $T \subseteq W_i$ with $|T| \leq m - m' = \nu m$. We will divide this range of $t := |T|$ into two parts and prove the following two statements:

(I) a.a.s. every $T \subset W_i$, $\frac{100}{p} \leq t \leq \nu\, |W_i|$, satisfies $|N(T)| \geq t$;

(II) a.a.s. every $T \subset W_i$, $t \leq \frac{100}{p}$, satisfies $|N(T)| \geq t$.

Proof of (I). Let $Y_i = \{y_1, y_2, \ldots, y_m\}$. Consider a fixed set $T \subset W_i$ with $100/p \leq t = |T| \leq \nu\, |W_i|$. We will be partially revealing π by exposing $\pi(H(y_k))$ one step at a time for $k = 1, 2, \ldots, m$. For convenience, set $H_k := H(y_k)$. Notice that if $\pi(H_k) \subseteq G(w)$ for some $w \in T$, then $(y_k, w) \in A^{\pi}_i$ and thus $y_k \in N(T)$.

Suppose that $\pi(H_j)$ has been exposed for all $j < k$. The set $\pi(H_k)$ is then a uniformly chosen d-subset of $U_k = U \setminus \bigcup_{j<k} \pi(H_j)$ (see Figure 2). We have

$$|U_k| \geq n - md \geq n - \frac{n}{d'} \geq \frac{n}{2}.$$

Therefore, by property (ii) from Lemma 5,

$$\left| \left\{ w \in T \,:\, |G(w) \cap U_k| \geq \frac{1}{2} p\, |U_k| > \frac{pn}{4} \right\} \right| \geq t - \frac{20}{p} \geq 0.8t. \tag{9}$$

Let

$$\mathcal{A}_k = \bigcup_{w \in T} \binom{G(w) \cap U_k}{d}.$$

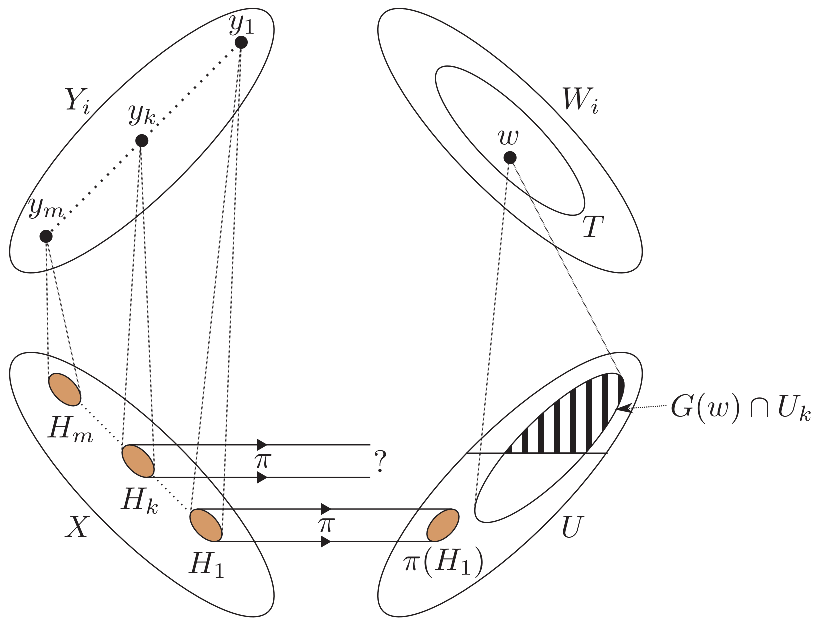

FIG. 2. *Illustration to the proof of Lemma* 8*, case* (I).

Note that $y_k \in N(T)$ iff $\pi(H_k) \in \mathcal{A}_k$. We are going to subdivide the range of t even further and assume first that $t \leq \frac{1}{2(12p)^d}$.

CLAIM 9. *For all* $k = 1, 2, \ldots, m$ *and* $t \leq \frac{1}{2(12p)^d}$*, we have*

$$|\mathcal{A}_k| \geq Q \overset{def.}{=} \frac{t}{2}\binom{pn/4}{d}.$$

Proof. By Bonferroni's inequality, we have

$$|\mathcal{A}_k| \geq \sum_{w \in T} \binom{|G(w) \cap U_k|}{d} - \sum_{w \neq w' \in T} \binom{|G(w) \cap G(w') \cap U_k|}{d}.$$

From (9) we conclude that

$$\sum_{w \in T} \binom{|G(w) \cap U_k|}{d} \geq 0.8t\binom{pn/4}{d}.$$

On the other hand, using property (i) from Lemma 5 applied to sets with two elements, we have

$$|G(w) \cap G(w')| \leq (1+\nu)p^2 n \leq (3/e)p^2 n$$

for every $w \neq w'$. Therefore, using the standard estimates $(M/l)^l \leq \binom{M}{l} \leq (eM/l)^l$,

$$\sum_{w \neq w' \in T} \binom{|G(w) \cap G(w') \cap U_k|}{d} \leq \binom{t}{2}\binom{(3/e)p^2 n}{d}$$
$$\leq \frac{1}{2}t^2\left(\frac{3p^2 n}{d}\right)^d \leq \frac{1}{2}t^2(12p)^d\binom{pn/4}{d}.$$

Under the assumption that $t \leq \frac{1}{2(12p)^d}$, the above inequalities imply the claim. ∎

For every $k = 1, 2, \ldots, m$, let $\mathcal{B}_k \subseteq \mathcal{A}_k$ be a fixed set with exactly Q elements (for concreteness, take the lexicographically first Q sets of $\mathcal{A}_k$). Further, define

$$I_k = \mathbb{I}[\pi(H_k) \in \mathcal{A}_k] \text{ and } J_k = \mathbb{I}[\pi(H_k) \in \mathcal{B}_k].$$

Let

$$Z_T = \sum_{k=1}^{m} I_k \text{ and } Z'_T = \sum_{k=1}^{m} J_k.$$

Observe that since $I_k \geq J_k$ for all k,

$$Z_T = |N(T)| \geq Z'_T.$$

It is easy to see that the variables J_k are independent and hence Z'_T is a generalized binomial random variable with mean

$$\mu'_T = \mathbb{E}[Z'_T] = \sum_{k=1}^{m} \frac{Q}{\binom{|U_k|}{d}}.$$

By Claim 9 we bound

$$\mu'_T \geq mQ\binom{n}{d}^{-1} \geq \frac{p^d m t}{2(4e)^d} \geq \frac{C^d \gamma t \log n}{2(4e)^d} \geq 16t \log n \tag{10}$$

by our choice of C.

Applying Chernoff's bound [15, Theorem 2.8] to Z'_T yields

$$\mathbb{P}\big[Z'_T \leq \mu'_T/2\big] \leq \exp\{-\mu_T/8\} \leq n^{-2t}.$$

Therefore, by the union bound

$$\begin{aligned} &\mathbb{P}\left[\text{there exists } T, \frac{100}{p} \leq t \leq \frac{1}{2(12p)^d} \ : \ Z'_T \leq \frac{\mu'_T}{2}\right] \\ &\qquad\qquad \leq \sum_{t=100/p}^{1/[2(12p)^d]} \binom{m}{t} n^{-2t} \leq \sum_{t=1}^{m} n^{-t} = o(1). \end{aligned} \tag{11}$$

Hence, a.a.s. every T with $\frac{100}{p} \leq t \leq \frac{1}{2(12p)^d}$ satisfies

$$|N(T)| \geq Z'_t \geq \frac{1}{2}\mu'_T \geq 8t \log n \geq t.$$

Consider now a set T with $\frac{1}{2(12p)^d} \leq t \leq \nu m$. Let $T_0 \subset T$ be an arbitrary set with cardinality $\frac{1}{2(12p)^d}$. We have

$$|N(T)| \geq |N(T_0)| = Z_{T_0} \geq Z'_{T_0} \geq \frac{\mu'_{T_0}}{2} \overset{(10)}{\geq} \frac{p^d m}{4(4e)^d}|T_0| = \nu m.$$

It follows that a.a.s. every T with $\frac{100}{p} \leq t \leq \nu m$ satisfies $|N(T)| \geq t$ and thus (I) is proved.

Proof of (II). Since in this case the sets T are small (i.e., $t = |T| < 100/p$), we will change our strategy and consider the inverse of the random mapping π. Recall that $N(T)$ has been defined in (8) and γ was defined in (6). Our aim will be to prove that a.a.s.

$$|N(T)| \geq p^d \frac{\gamma n}{2 \cdot 4^d} \cdot t \quad \text{for } \textit{all} \quad T \subset W_i\ ,\ t = |T| \leq \frac{\gamma}{p}. \tag{12}$$

This will also imply that a.a.s. all sets T with $\frac{\gamma}{p} < t \leq \frac{100}{p}$ satisfy

$$|N(T)| \geq p^d \frac{\gamma n}{2 \cdot 4^d} \cdot \frac{\gamma}{p} \geq \frac{100}{p} \geq t.$$

To prove (12), we fix a set $T = \{w_1, \dots, w_t\} \subset W_i\ ,\ t \leq \frac{\gamma}{p}$, and begin by constructing a disjoint family $\mathcal{N} = \{N_k \subseteq G(w_k) : k = 1, \dots, t\}$ such that $|N_k| = pn/2$ for every $w \in T$.

Claim 10. *There is a disjoint family $\mathcal{N} = \{N_k \subseteq G(w_k)\ :\ k = 1, \dots, t\}$ such that $|N_k| = pn/2$ for every $k = 1, \dots, t$.*

Proof. We will construct the desired family using a simple matching argument. A folklore corollary of Hall's theorem states that if for some integer s

$$\left| \bigcup_{w \in T'} G(w) \right| \geq s\, |T'|$$

for every $T' \subseteq T$, then there exists in G a star-matching saturating T, that is, a forest whose components are stars with s arms and centers at every $w \in T$.

For any $T' \subseteq T$, property (i) from Lemma 5 and Bonferroni's inequality yield that

$$\begin{aligned} \left| \bigcup_{w \in T'} G(w) \right| &\geq \sum_{w \in T'} |G(w)| - \sum_{w \neq w' \in T'} |G(w) \cap G(w')| \\ &\geq |T'|(1-\nu)pn - |T'|^2(1+\nu)p^2 n \\ &\geq \frac{1}{2} pn\, |T'|, \end{aligned} \tag{13}$$

where the third inequality holds by our assumption on t. The existence of family $\mathcal{N}$ follows from (13) and the above mentioned corollary of Hall's theorem. ∎

We will estimate $|N(T)|$ from below by counting how many elements $y \in Y_i$ are such that for some $k = 1, \dots, t$ we have $H(y) \subseteq \pi^{-1}(N_k)$. Indeed, this containment implies that

$$\pi(H(y)) \subseteq N_k \subseteq G(w_k),$$

which by (7) means that $(y, w_k) \in E(A_i^\pi)$ and thus $y \in N(T)$.

For $k = 1, \dots, t$ set

$$R_k = |\{y \in Y_k\ :\ H(y) \subseteq \pi^{-1}(N_k)\}|.$$

Further, let $R^T = R = \sum_{k=1}^t R_k$. Since the family $\mathcal{N}$ is disjoint, $|N(T)| \geq R$.

Claim 11. *For a sufficiently large constant C, we have*

$$\mathbb{P}\Big[R^T < \frac{t}{2}\Big(\frac{p}{4}\Big)^d m\Big] \leq n^{-2t}. \tag{14}$$

Observe that (12) follows directly from Claim 11 and the union bound. Indeed,

$$\mathbb{P}\Big[\exists\, T \,:\, t = |T| \leq \gamma/p \text{ and } R^T < \frac{t}{2}\Big(\frac{p}{4}\Big)^d m\Big]$$
$$\leq \sum_{t=1}^{\gamma/p} \binom{m}{t} n^{-2t} < \sum_t n^{-t} = o(1).$$

Therefore (II) will be established after we prove Claim 11.

Proof. Let $I_k = \mathbb{I}\big[R_k \geq \big(\frac{p}{4}\big)^d m\big]$ for every k, and let $Z = \sum_{k=1}^t I_k$. Clearly,

$$\mathbb{P}\Big[R < \frac{t}{2}\Big(\frac{p}{4}\Big)^d m\Big] \leq \mathbb{P}[Z < t/2].$$

For any $a \in \{0,1\}^t$ we have

$$\mathbb{P}[I_k = a_k \text{ for all } k] = \prod_{k=1} \mathbb{P}\big[I_k = a_k \,\big|\, I_1 = a_1, \ldots, I_{k-1} = a_{k-1}\big]. \tag{15}$$

We will show that

$$\mathbb{P}\big[I_k = 0 \,\big|\, I_1 = a_1, \ldots, I_{k-1} = a_{k-1}\big] \leq n^{-6} \tag{16}$$

regardless of the values $a_1, a_2, \ldots, a_{k-1}$. Let $|a| = \sum_{k=1}^t a_k$. In view of (15) and (16),

$$\begin{aligned}\mathbb{P}[Z < t/2] &= \sum_{a\,:\,|a|<t/2} \mathbb{P}[I_k = a_k \text{ for all } k] \\ &\leq \sum_{a\,:\,|a|<t/2} n^{-6\,(t-|a|)} \leq 2^t n^{-3t} < n^{-2t}.\end{aligned} \tag{17}$$

This proves that (16) implies (14). It remains to show (16).

To this end, the inverse of the random map π will be exposed in steps by revealing $\pi^{-1}(N_k)$ one at a time. For $1 \leq k \leq t$, let

$$X_k = X \setminus \bigcup_{j=1}^{k-1} \pi^{-1}(N_j) \text{ and } F_k = \{y \in Y_i \,:\, H(y) \subset X_k\}. \tag{18}$$

By the definition of F_k, for every $y \in Y_i \setminus F_k$ we have $H(y) \cap (X \setminus X_k) \neq \emptyset$. Since by construction the family $\{H(y) \,:\, y \in Y_i\}$ is disjoint, it follows that

$$|Y_i \setminus F_k| \leq |X \setminus X_k| = (k-1)\frac{pn}{2} \leq t\frac{pn}{2} \leq \frac{\gamma n}{2} \leq \frac{m}{2},$$

and thus $|F_k| \geq m/2$ (see Figure 3).

Suppose that $\pi^{-1}(N_j)$ has been exposed for all $1 \leq j \leq k-1$. In particular, $I_1, I_2, \ldots, I_{k-1}$ are determined. We need to compute the probability that $I_k = 0$ conditional on the exposed part of π. In this conditional space, the set $\pi^{-1}(N_k)$ is uniformly chosen among all $\frac{pn}{2}$-subsets of X_k.

It will be convenient to switch to a different model where independent choices are made for each vertex of X_k. Formally, consider a process which selects vertices of X_k independently with probability

$$q = \frac{pn}{2\,|X_k|}$$

to form a random subset of X_k denoted by $(X_k)_q$.

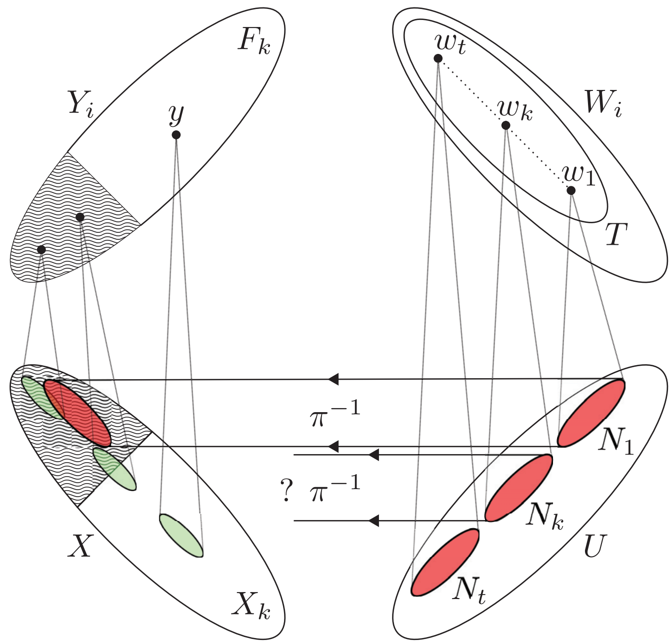

FIG. 3. *Illustration to the proof of Lemma* 8, *case* (II).

To link the two random models, we use Pittel's inequality [15, p. 17]. Let

$$\mathcal{Q} = \left\{ S \subseteq X_k \, : |\{y \in Y_k \, : \, H(y) \subseteq S\}| \geq \left(\frac{p}{4}\right)^d m \right\}$$

and notice that $R_k < \left(\frac{p}{4}\right)^d m$ is equivalent to $\pi^{-1}(N_k) \notin \mathcal{Q}$. Pittel's inequality then yields

$$\mathbb{P}\Big[I_k = 0\Big] = \mathbb{P}\Big[\pi^{-1}(N_k) \notin \mathcal{Q}\Big] \leq 3\sqrt{\frac{pn}{2}} \cdot \mathbb{P}\big[(X_k)_q \notin \mathcal{Q}\big],$$

where all probabilities are conditional upon $\pi^{-1}(N_1), \ldots, \pi^{-1}(N_{k-1})$.

Let $Q_k = |\{y \in Y_k : H(y) \subseteq (X_k)_q\}|$. Observe that Q_k has a binomial distribution with parameters $|F_k|$ and q^d and thus with mean

$$\mu_k := |F_k| \, q^d \geq \frac{m}{2}\left(\frac{p}{2}\right)^d$$

since $q \geq \frac{p}{2}$. Indeed, Q_k is the sum of indicator random variables $\mathbb{I}[H(y) \subseteq (X_k)_q]$, where $y \in F_k$. The independence of the variables stems from the fact that $\{H(y) : y \in Y_i\}$ is a disjoint family.

Finally, Chernoff's inequality [15, Theorem 2.1] yields, for $d \geq 2$,

$$\mathbb{P}\big[(X_k)_q \notin \mathcal{Q}\big] = \mathbb{P}\Big[Q_k < \left(\frac{p}{4}\right)^d m\Big] \leq \mathbb{P}\Big[Q_k < \frac{1}{2}\mu_k\Big] \leq \exp\Big\{-\frac{\gamma n p^d}{2^{d+4}}\Big\} \leq n^{-7}$$

by our choice of C and the fact that $m \geq \gamma n$. Therefore, (16) holds and the claim is proved. $\square$

We have proved that both (I) and (II) hold and therefore Lemma 8 and consequently Theorem 7 follow. $\square$

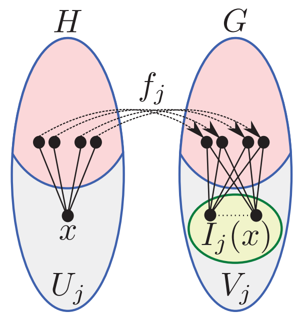

FIG. 4. *An illustration of the definition given by* (19).

4. An embedding scheme for bounded degree graphs. In this section we prove Theorem 2 by providing a scheme that embeds any graph H with $\Delta(H) \leq d$ and $|V(H)| = n$ into any given graph G satisfying properties (i)–(iii) from Lemma 6. Throughout the proof we assume that $d \geq 2$.

The embedding is done in two phases. It starts by embedding one vertex at a time until almost all the vertices of the graph are embedded. The rest of the graph is embedded by finding a perfect matching in some auxiliary graph. The first phase is greedy (it never regrets a decision) but takes into consideration a few invariants that guarantee that the embedding of the whole graph can be done. This structure is quite similar to [17]. However, several differences and subtleties are inherent to the sparse random graph case.

In the first phase we construct a sequence of partial embeddings $f_0, f_1, f_2, \ldots, f_k$ for some $k \geq n - \frac{n}{d^2+1}$, where each embedding extends the previous by one vertex. In the second phase all the remaining vertices are embedded in a single step.

Let G be a fixed graph satisfying properties (i)–(iii) from Lemma 6 with $\varepsilon = \varepsilon(d)$ sufficiently small. Fix a graph H with $\Delta(H) \leq d$ and n vertices. Label the vertices of H using the elements in $[n] = \{1, 2, \ldots, n\}$ in such a way that for $m = n - n/(d^2+1)$, the labels $\{m+1, m+2 \ldots, n\}$ are assigned to 2-*independent* vertices, that is, every two vertices are at distance at least 3 from each other. This labeling is indeed possible since the graph $J = H \cup H^2$ has degrees bounded by d^2. By Brooks's theorem, there is a proper (d^2+1)-coloring of J. Label some elements in the largest color class with $m+1, m+2, \ldots, n$. By construction, vertices with the same color are at distance at least 3 from each other.

Before we describe the embedding, we introduce some notation. Denote by U_j the set of vertices of H which are not embedded by f_j. Similarly, let V_j be the set of vertices in G which are not in the image of f_j. Define I_j to be a bipartite graph with classes (U_j, V_j), where for each $x \in U_j$ the neighborhood of x,

$$I_j(x) = G^{\cap}\Big(f_j\big(H(x) \setminus U_j\big)\Big) \cap V_j, \tag{19}$$

consists of all "candidates" for $f_{j+1}(x)$. More precisely, the set $I_j(x)$ consists of all elements $v \in V_j$ such that mapping x to v produces a valid extension of f_j. Indeed, in order for the edges incident to x in H to be preserved under the extension, the image of x must be adjacent (in G) to all vertices in $f_j(H(x) \setminus U_j)$. See Figure 4 for an illustration of the candidate set's definition.

In view of (19), the neighborhood of a vertex $v \in V_j$ is completely determined. Indeed, $x \in I_j(v)$ iff $v \in I_j(x)$, which means that $v \in G^{\cap}\big(f_j(H(x) \setminus U_j)\big)$. Consequently, one must have $f_j(H(x) \setminus U_j) \subset G(v)$. In particular, for every $v \in V_j$,

$$I_j(v) = \{x \in U_j \ : \ f_j(H(x) \setminus U_j) \subset G(v)\}. \tag{20}$$

The aim of the first phase is to produce an embedding f_k which embeds enough vertices of H so that $U_k \subset \{m+1, \ldots, n\}$ is 2-independent. The fact that the vertices in U_k are independent in H implies that their images may be chosen independently (they just need to be distinct for each vertex).

In the second phase we will find a perfect matching in I_k which will define the extension of f_k into a complete embedding of H into G.

4.1. Phase 1. In this section we introduce an induction hypothesis which is maintained for each f_j, $j = 0, 1, \ldots, k$. The induction step (embedding extension) is introduced in section 4.1.1. The induction is formally proved in section 4.1.3.

Induction Hypothesis. For every $x \in U_j$ we have

$$|I_j(x)| \geq c_j(x) \stackrel{\text{def.}}{=} \left(\frac{p}{4}\right)^{|H(x)\setminus U_j|} \frac{n}{4d^2}. \tag{21}$$

Moreover, for every $v \in V_j$, we have

$$|I_j(v)| \geq p^d \frac{n}{8d^5} \quad \text{ and } \quad |G(v) \cap V_j| \geq \frac{pn}{4d^2}. \tag{22}$$

The embedding f_0 is an empty map and since $G^\cap(\emptyset) = V(G)$, it follows that $I_0 = K(U_0, V_0)$, the complete bipartite graph with classes $U_0 = V(H)$ and $V_0 = V(G)$. It is clear that (21) and the first part of (22) are satisfied for $j = 0$. Moreover, since $G(v) \cap V_0 = G(v)$, property (i) from Lemma 6 implies that the second condition of (22) holds as well. In particular, the induction hypothesis is true for the base case $j = 0$.

Let us now consider how the auxiliary graph I_{j+1} evolves from I_j. Suppose that f_{j+1} extends f_j by mapping $x_{j+1} \mapsto v_{j+1}$. For any $x \in U_{j+1}$, the *candidate set* $I_{j+1}(x)$ satisfies

$$I_{j+1}(x) = \begin{cases} I_j(x) \setminus \{v_{j+1}\} & \text{if } x \notin H(x_{j+1}), \\ I_j(x) \cap G(v_{j+1}) & \text{if } x \in H(x_{j+1}). \end{cases} \tag{23}$$

Indeed, when $x \notin H(x_{j+1})$ we have $f_{j+1}(H(x)\setminus U_{j+1}) = f_j(H(x)\setminus U_j)$ and since $V_{j+1} = V_j \setminus \{v_{j+1}\}$, we infer by (19) that $I_{j+1}(x) = I_j(x) \setminus \{v_{j+1}\}$. On the other hand, if $x \in H(x_{j+1})$, then $f_{j+1}(H(x)\setminus U_{j+1}) = f_j(H(x)\setminus U_j) \cup \{v_{j+1}\}$ and consequently $I_{j+1}(x) = I_j(x) \cap G(v_{j+1})$.

Observe that every vertex $v \in V_{j+1}$ satisfies

$$I_{j+1}(v) = \begin{cases} I_j(v) \setminus \{x_{j+1}\} & \text{if } v \in G(v_{j+1}), \\ I_j(v) \setminus \big(\{x_{j+1}\} \cup H(x_{j+1})\big) & \text{if } v \notin G(v_{j+1}). \end{cases} \tag{24}$$

Indeed, for every $x \in I_j(v)$ with $x \in H(x_{j+1})$ we infer by (23) that $x \in I_{j+1}(v)$ iff $v \in G(v_{j+1})$. In the case $x \in I_j(v)$ with $x \notin H(x_{j+1})$, $x \neq x_{j+1}$, we have $x \in I_{j+1}(v)$.

Notice that a vertex v may lose at most $d + 1$ neighbors after a single vertex extension. Therefore, the only neighborhoods that ever shrink considerably are the neighborhoods of vertices in $H(x_{j+1}) \cap U_{j+1}$.

4.1.1. Induction step: Extending the embedding. Here we describe how the embeddings are extended. We postpone the proof of the induction step to section 4.1.3. A succinct description of the embedding scheme is stated as Algorithm 1.

Suppose that the partial embedding f_j has been constructed (recall that f_0 is an empty embedding) and satisfies the induction hypothesis. If the set U_j is 2-independent, we immediately end the first phase and execute Phase 2.

In each extension, we will look for vertices which are dangerously close to failing the induction hypothesis. For this, we distinguish three cases in the extension. We say that an extension is

H-critical if there exists a vertex $x \in U_j$ such that $|I_j(x)| < 2c_j(x)$,

G-critical if there exists a vertex $v \in V_j$ for which either $|I_j(v)| < p^d n/(4d^5)$ or $|G(v) \cap V_j| < pn/(2d^2)$, and

normal otherwise.

In our analysis we will show that few extensions are critical (see Lemma 14). (If the conditions for both H- and G-critical extensions hold, we use the convention that the extension is H-critical.)

In an H-critical extension, we choose the vertex $x \in U_j$ satisfying $|I_j(x)| < 2c_j(x)$ with the smallest label to be embedded. We apply Lemma 12 to such x in order to obtain $v \in I_j(x)$ and extend f_j by setting $x \mapsto v$. In case the extension is normal, take $x \in U_j$ with the smallest label and use Lemma 12 to define the image of x.

If the extension is G-critical, we choose $v \in V_j$ to be any of the vertices satisfying either $|I_j(v)| < p^d n/(2d^2)$ or $|G(v) \cap V_j| < pn/(2d^2)$. We apply Lemma 13 to the chosen vertex v in order to find $x \in I_j(v)$ and extend the embedding f_j by setting $x \mapsto v$.

Lemma 12 asserts that for any vertex $x \in U_j$ there is a candidate $v \in I_j(x)$ which ensures that no candidate set shrinks too much (see (23)).

LEMMA 12. *For any $x \in U_j$ there exists $v \in I_j(x)$ such that*

$$|I_j(x') \cap G(v)| \geq \frac{p}{2}\,|I_j(x')| \geq \frac{p}{2} c_j(x')$$

for all $x' \in H(x) \cap U_j$.

Proof. We may assume that $H(x) \cap U_j \neq \emptyset$ since otherwise the lemma holds trivially. Let $x' \in H(x) \cap U_j$ and consider the sets $A = I_j(x'), B = I_j(x) \subset V(G)$. Notice that $|H(x) \setminus U_j|, |H(x') \setminus U_j| \leq d-1$ (since $x, x' \in U_j$ are neighbors). It follows by the induction assumption over f_j that

$$p\,|A|\,|B| \geq p\Big(\frac{p}{4}\Big)^{2d-2}\Big(\frac{n}{4d^2}\Big)^2 > 100\varepsilon^{-3} n.$$

Applying property (ii) from Lemma 6 to the sets A, B we conclude that all but at most $\varepsilon\,|B|$ vertices $v \in B = I_j(x)$ fail to satisfy $|I_j(x') \cap G(v)| \geq p\,|I_j(x')|/2$.

Repeating the same argument for every element in $H(x) \cap U_j$ shows that there are at most $\varepsilon\,|I_j(x)|\,|H(x) \cap U_j| \leq \varepsilon d\,|I_j(x)|$ vertices in $I_j(x)$ that fail to satisfy the conditions of the lemma. Because of our choice of $\varepsilon \ll 1/d$ the lemma is proved. $\square$

Lemma 13 asserts that for any $v \in V_j$ there is some $x \in I_j(v)$ such that extending the embedding by $x \mapsto v$ does not shrink any candidate set too much.

LEMMA 13. *For any $v \in V_j$ there exists $x \in I_j(v) \subset U_j$ such that $|I_j(x') \cap G(v)| \geq p\,c_j(x')$ for all $x' \in H(x) \cap U_j$.*

Proof. Suppose that the statement fails for some $v \in V_j$. In particular, for each vertex $x \in I_j(v)$ there is some *witness*

$$x' \in H(x) \cap U_j \quad \text{for which} \quad |I_j(x') \cap G(v)| < p\,c_j(x'). \tag{25}$$

We assume that the induction hypothesis holds for f_j and thus

$$|G(v) \cap V_j| \geq pn/(4d^2) \quad \text{and} \quad |I_j(v)| \geq p^d n/(8d^5). \tag{26}$$

Let W be the set of all witnesses for v. Since a vertex $x' \in W$ can only be a witness to a neighbor $x \in H(x')$, and there are $|I_j(v)|$ choices for x, we must have $|W| \geq |I_j(v)|/d$. Let

$$\mathcal{W} = \bigl\{ f_j\bigl(H(x') \setminus U_j\bigr) \,:\, x' \in W \bigr\}.$$

Observe that each witness x' has a neighbor in $I_j(v) \subset U_j$ and thus the sets in $\mathcal{W}$ have at most $d-1$ elements each. We claim that every witness x' must have a neighbor which was already embedded. Indeed, otherwise $H(x') \setminus U_j = \emptyset$ and, in view of (19) and (21), this implies that $I_j(x') = V_j$ and $c_j(x') = n/4d^2$, which, by the induction assumption, then implies

$$|I_j(x') \cap G(v)| = |G(v) \cap V_j| \overset{(26)}{\geq} pn/(4d^2) = p\, c_j(x'),$$

contradicting (25). We have thus shown that $\emptyset \notin \mathcal{W}$. We will now find a disjoint subfamily $\mathcal{X} \subset \mathcal{W}$ with

$$|\mathcal{X}| > \frac{|W|}{d^3} \geq \frac{|I_j(v)|}{d^4} \overset{(26)}{\geq} \frac{p^d n}{8d^9} \gg \sqrt{n}$$

in which every set has the same cardinality $1 \leq \ell \leq d-1$. For this, take $W^* \subset W \subset V(H)$ to be a maximal 2-independent set (with respect to H). The family $\mathcal{W}^* = \bigl\{ f_j\bigl(H(x') \setminus U_j\bigr) \;:\; x' \in W^* \bigr\}$ is disjoint by construction and, moreover, $|\mathcal{W}^*| = |W^*| \geq \frac{|W|}{d^2+1}$. By the pigeonhole principle, there is $1 \leq \ell \leq d-1$ such that at least $\frac{|W^*|}{d-1}$ sets of $\mathcal{W}^*$ have cardinality ℓ. Let $\mathcal{X} \subset \mathcal{W}^*$ be the family of all ℓ-sets of $\mathcal{W}^*$. Clearly, $|\mathcal{X}| \geq \frac{|W^*|}{d-1} \geq \frac{|W|}{(d-1)(d^2+1)} > \frac{|W|}{d^3}$.

Apply property (iii) from Lemma 6 to $T = G(v) \cap V_j$ and $\mathcal{X}$. By averaging, there exists some $X \in \mathcal{X}$ for which $\#\{w \in T : X \subset G(w)\} \geq (1-\varepsilon)p^\ell |T|$. This is equivalent to $|G^\cap(X) \cap T| \geq (1-\varepsilon)p^\ell |T|$. Let $x' \in W$ be such that $X = f_j(H(x') \setminus U_j)$. Notice that by (19), $I_j(x') = G^\cap(X) \cap V_j$ and hence $I_j(x') \cap G(v) = G^\cap(X) \cap V_j \cap G(v) = G^\cap(X) \cap T$. Since by (26) $|T| \geq \frac{pn}{4d^2}$, it follows that

$$|I_j(x') \cap G(v)| \geq (1-\varepsilon)p^\ell \frac{pn}{4d^2} \geq (1-\varepsilon)4p\left(\frac{p}{4}\right)^\ell \frac{n}{4d^2} > p\, c_j(x').$$

However this contradicts the fact that x' is a witness. ∎

4.1.2. Bounding the number of critical extensions. We now prove that most of the extensions are normal. An extension that is either G-critical or H-critical will simply be called critical. For the proof we do not need to assume that the induction hypothesis holds.

LEMMA 14. *There are less than* $2d^3\sqrt{n}$ *critical extensions during Phase* 1.

Proof. Suppose for the sake of contradiction that the Cth critical extension, where $C = 2d^3\sqrt{n}$, occurs when extending f_{J-1} to f_J. At each normal extension, the embedded vertex is the one with the smallest label among all the vertices which have not been embedded so far. In particular, all vertices with labels $\{1, 2, \ldots, J-C-1\}$ must have been embedded after f_{J-1} was constructed and thus $U_{J-1} \subset [J-C, n]$.

Observe that U_{J-1} is not 2-independent as otherwise Phase 1 would have ended before f_J was constructed. On the other hand, the set $\left[n-\frac{n}{d^2+1}+1, n\right]$ is 2-independent by our particular choice of labels for $V(H)$. Consequently,

$$J \leq n - \frac{n}{d^2+1} + C. \tag{27}$$

Since $|U_J| = |V_J| = n - J$, (27) implies that $|U_J| = |V_J| \geq \frac{n}{d^2+1} - C$.

The lemma follows immediately from Claims 15 and 16 which bound the number of H- and G-critical extensions respectively.

CLAIM 15. *The number of H-critical extensions before f_J is at most $d(d^2+1)\sqrt{n}$.*

Let $x_1, \ldots, x_h \in V(H)$ be the vertices which were embedded in H-critical extensions before f_J was constructed. Let $j_i < J$ be such that x_i was (first) embedded by f_{j_i}. By the definition of an H-critical extension, $|I_{j_i-1}(x_i)| < 2c_{j_i-1}(x_i)$. Let $X_i = f_{j_i-1}\big(H(x_i) \setminus U_{j_i-1}\big)$ and notice that by (19), $I_{j_i-1}(x_i) = G^{\cap}(X_i) \cap V_{j_i-1} \supseteq G^{\cap}(X_i) \cap V_J$. In particular

$$|G^{\cap}(X_i) \cap V_J| \leq |I_{j_i-1}(x_i)| < 2c_{j_i-1}(x_i) \overset{(21)}{=} 2\Big(\frac{p}{4}\Big)^{|X_i|} \frac{n}{4d^2}.$$

Notice that $X_i \neq \emptyset$ since otherwise the above inequality implies that $|V_J| < \frac{n}{2d^2}$ and this contradicts the fact that $|V_J| \geq \frac{n}{d^2+1} - C$.

Now we will construct a disjoint family $\mathcal{X} \subset \{X_i \,:\, i = 1, \ldots, h\}$, where all sets have the same cardinality ℓ, $1 \leq \ell \leq d$, and $|\mathcal{X}| \geq \frac{h}{d(d^2+1)}$. To this end, we first select a maximal set $I \subset [h]$ for which $\{x_i \,:\, i \in I\}$ is 2-independent. Then we take $\mathcal{X} \subset \mathcal{X}_I = \{X_i \,:\, i \in I\}$ to be a subfamily containing only the sets with the most frequent cardinality in $\mathcal{X}_I$. Since $\mathcal{X}_I$ is a disjoint family with $|I|$ nonempty sets, it is clear that

$$|\mathcal{X}| \geq \frac{|I|}{d} \geq \frac{h}{d(d^2+1)}. \tag{28}$$

We thus have a disjoint family $\mathcal{X}$ of ℓ-sets in $V(G) \setminus V_J$ such that for every $X \in \mathcal{X}$,

$$|G^{\cap}(X) \cap V_J| < 2\Big(\frac{p}{4}\Big)^{\ell} \frac{n}{4d^2} < \Big(\frac{p}{4}\Big)^{\ell} |V_J|.$$

Since $w \in G^{\cap}(X)$ iff $X \subset G(w)$, we obtain

$$\#\{(w, X) \in V_J \times \mathcal{X} \,:\, X \subset G(w)\} = \sum_{X \in \mathcal{X}} |G^{\cap}(X) \cap V_J| < (p/4)^{\ell} |V_J|\, |\mathcal{X}|.$$

Because $|V_J| \geq \frac{n}{d^2+1} - C \gg \sqrt{n}$, in view of property (iii) from Lemma 6, we conclude that $|\mathcal{X}| < \sqrt{n}$. Therefore, by (28) we have shown that $h \leq d(d^2+1)\sqrt{n}$, which establishes Claim 15.

CLAIM 16. *The number of G-critical extensions before f_J is at most $2\sqrt{n}$.*

By the definition of G-critical extensions, if a vertex $v \in V(G)$ is the cause of the G-critical extension from, say, f_ℓ to $f_{\ell+1}$ ($\ell < J$), then either

(I) $pn/(2d^2) > |G(v) \cap V_\ell| > |G(v) \cap V_J|$ or

(II) $|I_\ell(v)| < p^d n/(4d^5)$.

Let B be the set of all vertices $v \in V(G)$ which caused G-critical extensions before f_J because they satisfy (I). Since

$$(1-\varepsilon)p\,|V_J| \geq (1-\varepsilon)p\Big(\frac{n}{d^2+1} - C\Big) > \frac{pn}{2d^2},$$

every vertex $v \in B$ satisfies $|G(v) \cap V_J| < (1-\varepsilon)p\,|V_J|$. By property (ii) from Lemma 6 applied to $A = V_J$ and B, we must have

$$|B| < \frac{100\varepsilon^{-3}n}{p\,|V_J|} = O(p^{-1}) = o(\sqrt{n}).$$

Now consider any set $T = \{v_1, \dots, v_t\} \subset V(G)$, $t \le C$, of vertices that cause a G-critical extension before f_J because (II) holds. We will construct a disjoint family $\mathcal{X}$ and use property (iii) from Lemma 6 to show that T must have fewer than $\sqrt{n}$ elements. Together with the upper bound on the size of $|B|$, the claim follows.

For every $1 \le i \le t$, let $j_i < J$ be such that f_{j_i} is the first embedding in which v_i appears in the image. For such vertices, we have

$$\begin{aligned} p^d n/(4d^5) &> |I_{j_i-1}(v_i)| \\ &\overset{(20)}{=} \#\{x \in U_{j_i-1} \,:\, f_{j_i-1}(H(x) \setminus U_{j_i-1}) \subset G(v_i)\} \\ &\ge \#\{x \in U_J \,:\, f_J(H(x) \setminus U_J) \subset G(v_i)\}, \end{aligned} \tag{29}$$

where the last inequality follows since $U_J \subset U_{j_i-1}$ and therefore $H(x) \setminus U_{j_i-1} \subset H(x) \setminus U_J$. Moreover, we also conclude by (20) that every $x \in U_J$ with $H(x) \setminus U_J = \emptyset$ must be in $I_{j_i-1}(v_i)$. In other words, for every i,

$$\{x \in U_J \,:\, H(x) \setminus U_J = \emptyset\} \subset I_{j_i-1}(v_i).$$

It follows that all but at most $p^d n/(4d^5)$ vertices $x \in U_J$ are such that $H(x) \setminus U_J \neq \emptyset$.

Next we are going to construct a disjoint family $\mathcal{X} \subset \{f_J(H(x) \setminus U_J) \,:\, x \in U_J\}$ where

(a) each set has the same cardinality ℓ, $1 \le \ell \le d$,
(b) no set in $\mathcal{X}$ contains an element of T, and
(c) $|\mathcal{X}| \ge (|U_J| - p^d n - td)/(2d^3) > \frac{n}{3d^5}$.

Let $U'_J \subset U_J$ be the set of all vertices $x \in U_J$ for which $H(x) \setminus U_J \neq \emptyset$ and $f_J(H(x) \setminus U_J) \cap T = \emptyset$—equivalently, $x \notin H(f_J^{-1}(T))$. There are at most $p^d n/(4d^5) + |H(f_J^{-1}(T))| < p^d n + dt$ vertices in $U_J \setminus U'_J$. Let $U^*_J \subset U'_J$ be a maximal 2-independent subset of U'_J. Take $\mathcal{X} \subset \mathcal{X}^* = \{f_J(H(x) \setminus U_J) \,:\, x \in U^*_J\}$ to be a family containing all the sets having the most frequent cardinality in $\mathcal{X}^*$. Since $t \le C = 2d^3\sqrt{n}$, it is simple to check that such $\mathcal{X}$ is a disjoint family satisfying (a)–(c). By construction,

$$\begin{aligned} \#\{(v_i, X) \in T \times \mathcal{X} \,:\, X \subset G(v_i)\} &\le \sum_{i=1}^{t} \#\{x \in U^*_J \,:\, f_J(H(x) \setminus U_J) \subset G(v_i)\} \\ &\overset{(29)}{<} p^d \frac{n}{4d^5}|T| \overset{(c)}{\le} \frac{3}{4} p^\ell |T|\,|\mathcal{X}|. \end{aligned} \tag{30}$$

Since $|\mathcal{X}| \gg \sqrt{n}$, in view of property (iii) from Lemma 6, we conclude that $t = |T| < \sqrt{n}$. Since the number of G-critical extensions before f_J is at most $|B| + |T|$, the claim is proved.

Claims 15 and 16 contradict our assumption that there were C critical extensions before f_J thus proving the lemma. □

4.1.3. Proof of the induction step. Since f_0 is an empty embedding, by (19) the graph I_0 is a complete bipartite graph with classes $\big(U_0 = V(H), V_0 = V(G)\big)$. Moreover, property (i) from Lemma 6 ensures that every vertex $v \in V_0$ satisfies $|G(v) \cap V_0| = |G(v)| > pn/2$. Consequently, the induction hypothesis holds for f_0.

Suppose that the induction hypothesis holds for $f_0, f_1, \ldots, f_{j-1}$, $j \geq 1$. The hypothesis could fail for f_j either because (21) fails for some $x \in U_j$ or because (22) fails for some $v \in V_j$. Claims 17 and 18 imply that neither (21) nor (22) fails, thus verifying the induction step.

CLAIM 17. *There is no vertex $x \in U_j$ for which* (21) *fails.*

Suppose that there is $x \in U_j$ for which (21) fails, namely, $|I_j(x)| < c_j(x)$.

Let ℓ, $1 \leq \ell \leq j$, be the largest index such that f_ℓ extends $f_{\ell-1}$ by embedding a neighbor x^* of x. Such index exists as otherwise x would have no embedded neighbors and this would imply that $|I_j(x)| = |V_j| > c_j(x)$. By construction, there are only two ways in which a vertex of $V(H)$ is embedded. For x^* this means that either

(a) the image of x^* under f_ℓ was chosen using Lemma 12 or

(b) x^* was selected as the preimage $f_\ell^{-1}(v)$ of a vertex $v \in V(G)$ using Lemma 13.

In case (a), Lemma 12 (applied with $x \leftarrow x^*$, $j \leftarrow \ell - 1$) provides $v \in I_{\ell-1}(x^*)$ for which $f_\ell \colon x^* \mapsto v$. Lemma 12 together with (23) ensures that

$$|I_\ell(x')| = |I_{\ell-1}(x') \cap G(v)| \geq (p/2)c_{\ell-1}(x')$$

for all $x' \in H(x^*) \cap U_{\ell-1}$.

In case (b), similarly to (a), we use Lemma 13 and (23) to ensure that $|I_\ell(x')| = |I_{\ell-1}(x') \cap G(v)| \geq p\, c_{\ell-1}(x')$ for all $x' \in H(x^*) \cap U_{\ell-1}$. In particular, because $x \in H(x^*) \cap U_{\ell-1}$, the conclusions hold for $x' = x$ and thus in either case (a) or (b), we have

$$|I_\ell(x)| \geq \frac{p}{2} c_{\ell-1}(x) \overset{(21)}{\geq} 2c_\ell(x).$$

Moreover, since no neighbor of x was embedded after f_ℓ, by (21) we have $c_\ell(x) = c_{\ell+1}(x) = \cdots = c_j(x)$.

In view of (23), we conclude that $|I_r(x) \setminus I_{r+1}(x)| \leq 1$ for all $\ell \leq r \leq j-1$. Since $|I_\ell(x)| \geq 2c_j(x)$ and $|I_j(x)| < c_j(x)$, for some $\ell < r \leq j-1$ we have $2c_j(x) - 1 \leq |I_r(x)| < 2c_j(x)$. Consequently, $|I_r(x)|, \ldots, |I_j(x)| < 2c_j(x)$. The vertex x is a witness that every extension between the embeddings $f_r, f_{r+1}, \ldots, f_j$ is H-critical. Indeed, during each such extension, some vertex with label smaller than x satisfied the conditions for an H-critical extension.

Observe that

$$j - r \geq |I_r(x) \setminus I_j(x)| = |I_r(x)| - |I_j(x)| \geq (2c_j(x) - 1) - c_j(x) = c_j(x) - 1.$$

Consequently, our assumption that x fails (21) implies that at least $j - r \geq c_j(x) - 1 \gg 2d^3\sqrt{n}$ critical extensions occurred after f_r. This contradicts Lemma 14. Hence, no such $x \in U_j$ exists and the claim is proved.

CLAIM 18. *There is no $v \in V_j$ for which* (22) *fails.*

Suppose that (22) fails to hold for f_j because there is $v \in V_j$ which satisfies either $|I_j(v)| < p^d n/(8d^5)$ or $|G(v) \cap V_j| < pn/(4d^2)$.

It is clear from (24) that for every $\ell \leq j$,

$$|I_{\ell-1}(v)| \geq |I_\ell(v)| \geq |I_{\ell-1}(v)| - (d+1).$$

Moreover, $|V_{\ell-1} \setminus V_\ell| = 1$ for all ℓ. It follows that for $\ell \leq j$, we have

$$|I_\ell(v)| \leq |I_j(v)| + (d+1)(j - \ell) \tag{31}$$

and

$$|G(v) \cap V_\ell| \le |G(v) \cap V_j| + (j - \ell). \tag{32}$$

Let $\ell < j$ be the largest index for which the extension from f_ℓ to $f_{\ell+1}$ was normal. By the conditions for a G-critical extension, we have $|I_\ell(v)| \ge p^d n/(4d^5)$ and $|G(v) \cap V_\ell| \ge pn/(2d^2)$. From (31) and (32), we have

$$\begin{aligned} j - \ell &\ge \max\left\{\frac{|I_\ell(v)| - |I_j(v)|}{d+1}, |G(v) \cap V_\ell| - |G(v) \cap V_j|\right\} \\ &\ge \max\left\{\frac{p^d n/(4d^5) - |I_j(v)|}{d+1}, \frac{pn}{2d^2} - |G(v) \cap V_j|\right\}. \end{aligned} \tag{33}$$

If $|I_j(v)| < p^d n/(8d^5)$ we obtain $j - \ell \ge p^d n/[8d^5(d+1)]$ and if $|G(v) \cap V_j| < pn/(4d^2)$ we obtain $j - \ell \ge pn/(4d^2)$. Either way, the fact that v fails (22) and the definition of p imply that $j - \ell > p^d n/(16d^6) \gg 2d^3\sqrt{n}$. By the definition of ℓ, at least $j-\ell$ critical extensions occurred during the embedding process, which contradicts Lemma 14. Therefore no such $v \in V_j$ exists and the claim is established.

We have shown that the induction hypothesis must hold for f_j and therefore the proof of the induction is complete.

4.2. Phase 2. Suppose that f_k is the partial embedding constructed in Phase 1. The induction hypothesis ensures that

$$|I_k(x)| \ge c_k(x) \ge \left(\frac{p}{4}\right)^d \frac{n}{4d^2} > d\sqrt{n} \tag{34}$$

for all $x \in U_j$ and

$$|I_k(v)| \ge p^d \frac{n}{8d^5} > d\sqrt{n} \tag{35}$$

for all $v \in V_j$.

Moreover, by construction, the set U_k is 2-independent in H. Consequently, the family $\mathcal{F} = \{H(x) : x \in U_k\}$ is disjoint and each set $H(x)$ is contained in $V(H) \setminus U_k$. We claim that if there exists a perfect matching M in I_k, the extension f of f_k produced by mapping $x \in U_k$ to $v \in I_k(v)$ for all $(x, v) \in M$ is a valid embedding of H into G. The mapping f is clearly a bijection. Moreover, for every $e = xy \in E(H)$, with both $x, y \notin U_k$, the mapping f_k is such that $\{f_k(x), f_k(y)\} \in E(G)$. For $e = xy \in E(H)$ with $x \in U_k$ and $y \notin U_k$, we have $f(x) \in I_k(x) = G^\cap(f_k(H(x))) \cap V_k \subset G(f(y))$ and thus $\{f(x), f(y)\} \in E(G)$.

It remains to show that I_k contains a perfect matching. Set $m = |U_k| = |V_k|$ and assume that no perfect matching exists. Hall's theorem implies that there are sets $A \subset U_k$ and $B = V_k \setminus I_k(A)$ such that $|A| > |I_k(A)| = m - |B|$. This condition also implies that $I_k(B) \subset U_k \setminus A$ and thus $|I_k(B)| \le m - |A| < |B|$. Moreover, (34) and (35) imply that $|I_k(A)|, |I_k(B)| > d\sqrt{n}$ and thus $|A|, |B| > d\sqrt{n}$.

Consider a (disjoint) subfamily $\mathcal{X} \subset \{f_k(H(x)) : x \in A\}$ in which every set has the same cardinality and $|\mathcal{X}| \ge |A|/d > \sqrt{n}$. Given the fact that $|\mathcal{X}| > \sqrt{n}$ and $|B| > \sqrt{n}$, after applying property (iii) from Lemma 6 to $\mathcal{X}$ and $B \subset V_k$ we infer that there must exist a pair $(w, X) \in B \times \mathcal{X}$ with $X \subset G(w)$—equivalently, $w \in G^\cap(f_k(H(x)))$ for $x \in A$ such that $X = f_k(H(x))$. In view of (19), this means that there is an edge in I_k connecting $w \in B$ to $x \in A$. This contradicts the definition of B and therefore I_k must contain a perfect matching.

ALGORITHM 1. *Embedding graphs with bounded degree—Phase* 1.

Input : A graph H with n vertices and $\Delta(H) \leq d$ and a graph G satisfying Properties (i)-(iii) from Lemma 6.
Output: A partial embedding $f_k \colon V(H) \to V(G)$.

$j \leftarrow 0$;
while U_j *is not* 2*-independent* **do**
    if $\exists\, x \in U_j$ *such that* $|I_j(x)| < 2c_j(x)$ **then**
        // H-critical extension
        pick $x \in U_j$ satisfying $|I_j(x)| < 2c_j(x)$ with the smallest label ;
        pick $v \in I_j(x)$ to satisfy the conclusion of Lemma 12 ;
        set $f_{j+1} \colon x \mapsto v$;
    else if $\exists\, v \in V_j$ *such that* $|I_j(v)| < \frac{p^d n}{4d^5}$ *or* $|G(v) \cap V_j| < \frac{pn}{2d^2}$ **then**
        // G-critical extension
        let $v \in V_j$ be any such vertex ;
        pick $x \in I_j(v)$ to satisfy the conclusion of Lemma 13 ;
        set $f_{j+1} \colon x \mapsto v$;
    else
        // normal extension
        let $x \in U_j$ be the vertex with smallest label ;
        pick $v \in I_j(x)$ to satisfy the conclusion of Lemma 12 ;
        set $f_{j+1} \colon x \mapsto v$;
    $j \leftarrow j + 1$;

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