

AN EXPONENTIAL-TYPE UPPER BOUND
FOR FOLKMAN NUMBERS

VOJTĚCH RÖDL*, ANDRZEJ RUCIŃSKI†, MATHIAS SCHACHT‡

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For given integers k and r , the Folkman number $f(k; r)$ is the smallest number of vertices in a graph G which contains no clique on $k+1$ vertices, yet for every partition of its edges into r parts, some part contains a clique of order k . The existence (finiteness) of Folkman numbers was established by Folkman (1970) for $r=2$ and by Nešetřil and Rödl (1976) for arbitrary r , but these proofs led to very weak upper bounds on $f(k; r)$.

Recently, Conlon and Gowers and independently the authors obtained a doubly exponential bound on $f(k; 2)$. Here, we establish a further improvement by showing an upper bound on $f(k; r)$ which is exponential in a polynomial of k and r . This is comparable to the known lower bound $2^{\Omega(rk)}$. Our proof relies on a recent result of Saxton and Thomason (or, alternatively, on a recent result of Balogh, Morris, and Samotij) from which we deduce a quantitative version of Ramsey's theorem in random graphs.

1. Introduction

For two graphs, G and F , and an integer $r \geq 2$ we write $G \rightarrow (F)_r$ if every r -coloring of the edges of G results in a monochromatic copy of F . By a copy we mean here a subgraph of G isomorphic to F . Let K_k stand for the complete graph on k vertices and let $R(k; r)$ be the r -color Ramsey number,

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that is, the smallest integer n such that $K_n \rightarrow (K_k)_r$. As it is customary, we suppress $r=2$ and write $R(k) := R(k; 2)$ as well as $G \rightarrow F$ for $G \rightarrow (F)_2$.

In 1967 Erdős and Hajnal [8] asked if for some l , $k + 1 \leq l < R(k)$, there exists a graph G such that $G \rightarrow K_k$ and $G \not\rightarrow K_l$. Graham [12] answered this question in positive for $k=3$ and $l=6$ (with a graph on eight vertices), and Pósa (unpublished) for $k=3$ and $l=5$. Folkman [10] proved, by an explicit construction, that such a graph exists for every $k \geq 3$ and $l = k + 1$. He also raised the question to extend his result for more than two colors, since his construction was bound to two colors.

For integers k and r , a graph G is called $(k; r)$ -Folkman if $G \rightarrow (K_k)_r$ and $G \not\rightarrow K_{k+1}$. We define the r -color Folkman number for K_k by

$$f(k; r) = \min\{n \in \mathbb{N} : \exists G \text{ such that } |V(G)| = n \text{ and } G \text{ is } (k; r)\text{-Folkman}\}.$$

For $r = 2$ we set $f(k) := f(k; 2)$. It follows from [10] that $f(k)$ is well defined for every integer k , i.e., $f(k) < \infty$. This was extended by Nešetřil and Rödl [17], who showed that $f(k; r) < \infty$ for an arbitrary number of colors r .

Already the determination of $f(3)$ is a difficult, open problem. In 1975, Erdős [7] offered max(100 dollars, 300 Swiss francs) for a proof or disproof of $f(3) < 10^{10}$. For the history of improvements of this bound see [5], where a computer assisted construction is given yielding $f(3) < 1000$. For general k , the only previously known upper bounds on $f(k)$ come from the constructive proofs in [10] and [17]. However, these bounds are tower functions of height polynomial in k . On the other hand, since $f(k) \geq R(k)$, it follows by the well known lower bound on the Ramsey number that $f(k) \geq 2^{k/2}$, which for $k=3$ was improved to $f(3) \geq 19$ [19].

We prove an upper bound on $f(k; r)$ which is exponential in a polynomial of k and r . Set $R := R(k; r)$ for the r -color Ramsey number for K_k . It is known that there exists some $c > 0$ such that for every $r \geq 2$ and $k \geq 3$ we have

$$2^{crk} < R < r^{rk}.$$

The upper bound already appeared in the work of Skolem [25]. The lower bound obtained from a random r -coloring of the complete graphs is of the form $r^{k/2}$. However, Lefmann [14] noted that the simple inequality $R(k; s + t) \geq (R(k; s) - 1)(R(k; t) - 1) + 1$ yields a lower bound of the form $2^{kr/4}$. Using iteratively random 3-colorings in this “product-type” construction yields a slightly better lower bound of the form $3^{rk/6}$. Our main result establishes an upper bound on the Folkman number $f(k; r)$ of similar order of magnitude.

Theorem 1. *For all integers $r \geq 2$ and $k \geq 3$,*

$$f(k; r) \leq k^{400k^4} R^{40k^2} \leq 2^{c(k^4 \log k + k^3 r \log r)},$$

for some $c > 0$ independent of r and k .

To prove Theorem 1, we consider a random graph $G(n, p)$, $p = Cn^{-\frac{2}{k+1}}$, where $n = n(k, r)$ and $C = C(n, k, r)$ and carefully estimate from below the probabilities $\mathbb{P}(G(n, p) \rightarrow (K_k)_r)$ and $\mathbb{P}(G(n, p) \not\rightarrow K_{k+1})$, so that their sum is strictly greater than 1. The latter probability is easily bounded by the FKG inequality. However, to set a bound on $\mathbb{P}(G(n, p) \rightarrow (K_k)_r)$ we rely on a recent general result of Saxton and Thomason [24], elaborating on ideas of Nenadov and Steger [15] (see Remark 3).

Remark 1. Instead of the Saxton-Thomason theorem, we could have used a concurrent result of Balogh, Morris, and Samotij [1], which, by using our method, yields only a slightly worse upper bound on the Folkman numbers $f(k; r)$ than Theorem 1 (the k^4 in the exponent has to be replaced k^6).

Remark 2. In a related paper [23], we combined ideas from [9,20,22] and, for $r = 2$, obtained another proof of the Ramsey threshold theorem that yields a self-contained derivation of a double-exponential bound for the two-color Folkman numbers $f(k)$. Independently, a similar double-exponential bound for $f(k; r)$, for $r \geq 2$, was obtained by Conlon and Gowers [2] by a different method.

Motivated by the original question of Erdős and Hajnal, one can also define, for $r = 2$, $k \geq 3$, and $k + 1 \leq l \leq R(k)$, a *relaxed Folkman number* as

$$f(k, l) = \min\{n : \exists G \text{ such that } |V(G)| = n, \quad G \rightarrow K_k \text{ and } G \not\rightarrow K_l\}.$$

Note that $f(k, k+1) = f(k)$. As mentioned above, Graham [12] found out that $f(3, 6) = 8$, while Nenov [16] and Piwakowski, Radziszowski and Urbański [18] determined that $f(3, 5) = 15$ (see also [26]).

Of course, the problem is easier when the difference $l - k$ is bigger. Our final result provides an exponential bound of the form $f(k, l) \leq \exp\{-ck\}$, when l is close to but bigger than $4k$ (the constant c is proportional to the reciprocal of the difference between l/k and 4).

Theorem 2. For every $0 < \alpha < \frac{1}{4}$ there exists k_0 such that for k and l satisfying $k \geq k_0$ and $k \leq \alpha l$,

$$f(k; l) \leq 2^{4k/(1-4\alpha)}.$$

It would be interesting to decide if the true order of the logarithm of $f(k, k + 1) = f(k)$ is also linear in k .

The paper is organized as follows. In the next section we prove our main result, Theorem 1, while Theorem 2 is proved in Section 3. Finally, a short Section 4 offers a brief discussion of the analogous problem for hypergraphs.

Most logarithms in this paper are binary and are denoted by \log . Only occasionally, when citing a result from [24] (Theorem 5 in Section 2 below), we will use the natural logarithms, denoted by \ln .

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2. Proof of Theorem 1

We will prove Theorem 1 by the probabilistic method. Let $G(n, p)$ be the binomial random graph, where each of the $\binom{n}{2}$ possible edges is present, independently, with probability p . We are going to show that for every $n \geq k^{40k^4} R^{10k^2}$ and a suitable function $p = p(n)$, with positive probability, $G(n, p)$ has simultaneously two properties: $G(n, p) \rightarrow (K_k)_r$ and $G(n, p) \not\rightarrow K_{k+1}$. Of course, this will imply that there exists a $(k; r)$ -Folkman graph on n vertices. We begin with a simple lower bound on $\mathbb{P}(G(n, p) \not\rightarrow K_{k+1})$.

Lemma 3. *For all $k, n \geq 3$, and $C > 0$, if $p = Cn^{-2/(k+1)} \leq \frac{1}{2}$, then*

$$\mathbb{P}(G(n, p) \not\rightarrow K_{k+1}) > \exp(-C \binom{k+1}{2} n).$$

Proof. By applying the FKG inequality (see, e.g., [13, Theorem 2.12 and Corollary 2.13]), we obtain the bound

$$\begin{aligned} \mathbb{P}(G(n, p) \not\rightarrow K_{k+1}) &\geq \left(1 - p \binom{k+1}{2}\right)^{\binom{n}{k+1}} \\ &\geq \exp\left(-2C \binom{k+1}{2} n^{-k} \binom{n}{k+1}\right) > \exp\left(-C \binom{k+1}{2} n\right), \end{aligned}$$

where we also used the inequalities $\binom{n}{k+1} < n^{k+1}/2$ and $1 - x \geq e^{-2x}$ for $0 < x < \frac{1}{2}$. ■

The main ingredient of the proof of Theorem 1 traces back to a theorem from [20] establishing edge probability thresholds for Ramsey properties of $G(n, p)$. A special case of that result states that for all integers $k \geq 3$ and $r \geq 2$ there exists a constant C such that if $p = p(n) \geq Cn^{-\frac{2}{k+1}}$, then $\lim_{n \rightarrow \infty} \mathbb{P}(G(n, p) \rightarrow (K_k)_r) = 1$.

Adapting an idea of Nenadov and Steger [15] (see Remark 3 for more on that), and based on a result of Saxton and Thomason [24], we obtain the

following quantitative version of the above random graph theorem. Recall our notation $R = R(k; r)$ for the r -color Ramsey number and notice an easy lower bound

$$(1) \quad R(k; r) > 2r$$

valid for all $r \geq 2$ and $k \geq 3$ (just consider a factorization of K_{2r}).

Lemma 4. *For all integers $r \geq 2$, $k \geq 3$, and*

$$(2) \quad n \geq k^{400k^4} R^{40k^2},$$

the following holds. Set

$$(3) \quad b = \frac{1}{2R^2}, \quad C = 2^{5\sqrt{\log n \log k}} R^{16}, \quad \text{and} \quad p = Cn^{-\frac{2}{k+1}}.$$

Then

$$\mathbb{P}(G(n, p) \rightarrow (K_k)_r) \geq 1 - \exp(-bp\binom{n}{2}).$$

We devote the next two subsections to the proof of Lemma 4. Now, we deduce Theorem 1 from Lemmas 3 and 4.

Proof of Theorem 1. For given r and k , let n be as in (2), and let b, C , and p be as in (3). Below we will show that these parameters satisfy not only the assumptions of Lemma 4, but also the assumption $p \leq \frac{1}{2}$ of Lemma 3, as well as an additional inequality

$$(4) \quad n \geq (3/b)^{\frac{k+1}{k-1}} C^{\binom{k+2}{2}}.$$

With these two inequalities at hand, we may quickly finish the proof of Theorem 1. Indeed, (4) implies that

$$(5) \quad bp\binom{n}{2} \geq \frac{1}{3}bpn^2 = (b/3)Cn^{1+\frac{k-1}{k+1}} \stackrel{(4)}{\geq} C^{\binom{k+1}{2}}n,$$

which, by Lemma 3, implies in turn that

$$\mathbb{P}(G(n, p) \not\rightarrow K_{k+1}) > \exp(-bp\binom{n}{2}).$$

Since, by Lemma 4,

$$\mathbb{P}(G(n, p) \rightarrow (K_k)_r) \geq 1 - \exp(-bp\binom{n}{2}),$$

we conclude that

$$\mathbb{P}(G(n, p) \rightarrow (K_k)_r \text{ and } G(n, p) \not\rightarrow K_{k+1}) > 0.$$

Thus, there exists a $(k; r)$ -Folkman graph on n vertices, and thus, $f(k) \leq k^{400k^4} R^{40k^2}$.

It remains to show that $p \leq \frac{1}{2}$ and that (4) holds. The first inequality is equivalent to

$$(6) \quad n \geq (2C)^{\frac{k+1}{2}}.$$

We will now show that this inequality is a consequence of (4) and then establish (4) itself. Since $C > 2$ and $3/b \stackrel{(3)}{=} 6R^2 \geq 1$, we infer that

$$(3/b)^{\frac{k+1}{k-1}} C^{\binom{k+2}{2}} \geq C^{\binom{k+2}{2}} \geq (2C)^{\frac{k+1}{2}},$$

and hence, (6) indeed follows from (4).

Finally, we establish (4). In doing so we will use again the identity $3/b \stackrel{(3)}{=} 6R^2$, as well as the inequalities $36 \leq C$, which follows from (2) and (3), $\binom{k+2}{2} \leq k^2 + 1 \leq 2k^2 - 1$, and $\frac{k+1}{k-1} \leq 2$, valid for all $k \geq 3$. The R-H-S of (4) can be bounded from above by

$$(6R^2)^{\frac{k+1}{k-1}} C^{\binom{k+2}{2}} \leq 36R^4 C^{\binom{k+2}{2}} \leq R^4 C^{k^2+2} \leq 2^{10k^2 \sqrt{\log n \log k}} R^{20k^2}.$$

Hence, it suffices to show that

$$(7) \quad n \geq 2^{10k^2 \sqrt{\log n \log k}} R^{20k^2}.$$

Observe that, by (2), $\frac{1}{2} \log n \geq 20k^2 \log R$, and thus, it remains to check that

$$\frac{1}{2} \log n \geq 10k^2 \sqrt{\log n \log k},$$

or equivalently that

$$\log n \geq 400k^4 \log k.$$

This, however, follows trivially from (2). ▀

2.1. The proof of Lemma 4 – preparations

In this and the next subsection we present a proof of Lemma 4, which is inspired by the work of Nenadov and Steger [15] and is based on a recent general result of Saxton and Thomason [24] on the distribution of independent sets in hypergraphs. For a hypergraph H , a subset $I \subseteq V(H)$ is *independent* if the subhypergraph $H[I]$ induced by I in H has no edges.

For an h -graph H , the degree $d(J)$ of a set $J \subset V(H)$ is the number of edges of H containing J . (Since in our paper letter r is reserved for the

number of colors, we will use h for hypergraph uniformity.) We will write $d(v)$ for $d(\{v\})$, the ordinary vertex degree. We further define, for a vertex $v \in V(H)$ and $j=2, \dots, h$, the maximum j -degree of v as

$$d_j(v) = \max \left\{ d(J) : v \in J \subset \binom{V(H)}{j} \right\}.$$

Finally, the co-degree function of H with a formal variable τ is defined in [24] as

$$(8) \quad \delta(H, \tau) = \frac{2^{\binom{h}{2}-1}}{nd} \sum_{j=2}^h \frac{\sum_v d_j(v)}{2^{\binom{j-1}{2}} \tau^{j-1}},$$

where the inner sum is taken over all vertices $v \in V(H)$ and d is the average vertex degree in H , that is, $d = \frac{1}{n} \sum_v d(v)$.

Theorem 5 below is an abridged version of [24, Corollary 3.6], where we suppress part of conclusion (a) (about the sets T_i), as well as the “Moreover” part therein, since we do not use this additional information here. In part (c) of the theorem below, for convenience, we switch from \ln to \log , but only on the R-H-S of the upper bound on $\ln|\mathcal{C}|$.

Theorem 5 (Saxton & Thomason, [24]). *Let H be an h -graph on vertex set $[n]$ and let ε and τ be two real numbers such that $0 < \varepsilon < 1/2$,*

$$\tau \leq 1/(144(h!)^2 h) \quad \text{and} \quad \delta(H, \tau) \leq \varepsilon/(12(h!)).$$

Then there exists a collection \mathcal{C} of subsets of $[n]$ such that the following three properties hold.

- (a) *For every independent set I in H there exists a set $C \in \mathcal{C}$ such that $I \subset C$.*
- (b) *For all $C \in \mathcal{C}$, we have $e(H[C]) \leq \varepsilon e(H)$.*
- (c) *$\ln|\mathcal{C}| \leq c \log(1/\varepsilon) \tau \log(1/\tau) n$, where $c = 800(h!)^3 h$.*

We will now tailor the above result to our application. The hypergraphs we consider have a very symmetric structure. Given k and n , let $H(n, k)$ be the hypergraph with vertex set $\binom{[n]}{2}$, the edges of which correspond to all copies of K_k in the K_n with vertex set $[n]$. Thus, $H(n, k)$ has $\binom{n}{2}$ vertices, $\binom{n}{k}$ edges, and is $\binom{k}{2}$ -uniform and $\binom{n-2}{k-2}$ -regular.

For $J \subseteq \binom{[n]}{2}$, the degree of J in $H(n, k)$ is $d(J) = \binom{n-v_J}{k-v_J}$, where v_J is the number of vertices in J treated as a graph on $[n]$ rather than a subset of vertices of $H(n, k)$. Thus, over all J with $|J| = j$, $d(J)$ is maximized by

the smallest possible value of v_J , that is, when $v_J = l_j$, the smallest integer l such that $j \leq \binom{l}{2}$. Consequently, for every vertex v of $H(n, k)$ (that is, an edge of K_n on $[n]$) and for each $j = 2, \dots, \binom{k}{2}$, we have

$$d_j(v) = \binom{n - l_j}{k - l_j}.$$

Clearly, $l_j \geq 3$ for $j \geq 2$, which will be used later.

Let

$$\delta(n, k, \tau) := \sum_{j=2}^{\binom{k}{2}} \frac{2^{k^4} k^{k-2}}{\tau^{j-1} n^{l_j-2}}.$$

The co-degree function of $H(n, k)$ can be bounded by $\delta(n, k, \tau)$.

Claim 6.

$$\delta(H(n, k), \tau) \leq \delta(n, k, \tau).$$

Proof. By the definition of $\delta(H, \tau)$ in (8) with h replaced by $\binom{k}{2}$, n by $\binom{n}{2}$, d by $\binom{n-2}{k-2}$, $d_j(v)$ by $\binom{n-l_j}{k-l_j}$, and with $2^{\binom{j-1}{2}}$ dropped out from the denominator, we have

$$\delta(H(n, k), \tau) \leq 2^{k^4} \sum_{j=2}^{\binom{k}{2}} \frac{\binom{n-l_j}{k-l_j}}{\tau^{j-1} \binom{n-2}{k-2}}.$$

Now, observe that $\frac{\binom{n-l_j}{k-l_j}}{\binom{n-2}{k-2}} \leq (k/n)^{l_j-2}$ and $l_j \leq k$. ■

The most important property of hypergraph $H(n, k)$ is that a subset S of the vertices of H corresponds to a graph G with vertex set $[n]$ and edge set S , and S is an independent set in $H(n, k)$ if and only if the corresponding graph G is K_k -free. We now apply Theorem 5 to $H(n, k)$.

Corollary 7. *Let $k \geq 3$, $n \geq 3$, and let ϵ and τ be two real numbers such that $0 < \epsilon < 1/2$,*

$$(9) \quad \tau \leq (k^2!)^{-2} \quad \text{and} \quad \delta(n, k, \tau) \leq \frac{\epsilon}{k^2!}.$$

Then there exists a collection \mathcal{C} of subgraphs of K_n such that the following three properties hold.

- (a) *For every K_k -free graph $G \subseteq K_n$ there exists a graph $C \in \mathcal{C}$ such that $G \subset C$.*
- (b) *For all $C \in \mathcal{C}$, C contains at most $\epsilon \binom{n}{k}$ copies of K_k .*
- (c) *$\ln |\mathcal{C}| \leq (2k^2)! \log(1/\epsilon) \tau \log(1/\tau) \binom{n}{2}$.*

Proof. Note that for $k \geq 3$,

$$k^2! > 12\binom{k}{2}! \quad \text{and, consequently,} \quad (k^2!)^2 > 144\binom{k}{2}!\binom{k}{2},$$

and that, by Claim 6, $\delta(H(n, k), \tau) \leq \delta(n, k, \tau)$. Thus, the assumptions of Theorem 5 hold for $H := H(n, k)$ with $h = \binom{k}{2}$, and its conclusions (a-c) translate into the corresponding properties (a-c) of Corollary 7. Finally, notice that

$$(2k^2)! > c = 864 \left(\binom{k}{2}! \right)^3 \binom{k}{2}. \quad \blacksquare$$

In the next subsection we deduce Lemma 4 from Corollary 7. First, however, we make a simple observation about the number of monochromatic copies of K_k in every coloring of K_n . Recall that $R = R(k; r)$ is the r -color Ramsey number for K_k and set

$$(10) \quad \alpha = \binom{R}{k}^{-1}.$$

Proposition 8. *Let $n \geq R$. For every $(r + 1)$ -coloring of the edges of K_n either there are more than $\frac{\alpha}{2} \binom{n}{k}$ monochromatic copies of K_k colored by the first r colors, or more than $\frac{1}{R^2} \binom{n}{2}$ edges receive color $r + 1$.*

Proof. Consider an $(r + 1)$ -coloring of the edges of K_n . Let $x \binom{n}{R}$ be the number of the R -element subsets of the vertices of K_n with no edge colored by color $r + 1$. By the definition of R , each of these subsets induces in K_n a monochromatic copy of K_k . Thus, counting repetitions, there are at least

$$x \frac{\binom{n}{R}}{\binom{n-k}{R-k}} = x \frac{\binom{n}{k}}{\binom{R}{k}} = x\alpha \binom{n}{k}$$

monochromatic copies of K_k colored by one of the first r colors. Suppose that their number is at most

$$\frac{\alpha}{2} \binom{n}{k}.$$

Then $x \leq \frac{1}{2}$, that is, at least a half of the R -element subsets of $V(K_n)$ contain at least one edge colored by $r + 1$. Hence, color $r + 1$ appears on at least

$$\frac{\frac{1}{2} \binom{n}{R}}{\binom{n-2}{R-2}} = \frac{\frac{1}{2} \binom{n}{2}}{\binom{R}{2}} > \frac{1}{R^2} \binom{n}{2}$$

edges of K_n . This completes the proof. \blacksquare

2.2. Proof of Lemma 4 – details

Let $r \geq 2$, $k \geq 3$, and let n, b, C , and p be as in Lemma 4, see (3) and (2). We have to show that

$$\mathbb{P}(G(n, p) \rightarrow (K_k)_r) \geq 1 - \exp(-bp\binom{n}{2}).$$

First we set up a few auxiliary constants required for the application of Corollary 7. Recalling that α is defined in (10), let

$$(11) \quad \varepsilon = \frac{\alpha}{2r},$$

$$(12) \quad C_0 = 2^{4\sqrt{\log n}} R^{10/k}, \quad \text{and} \quad \tau = C_0 n^{-\frac{2}{k+1}}.$$

We will now prove that the above defined constants ε and τ satisfy the assumptions of Corollary 7.

Claim 9. *Inequalities (9) hold true for every $k \geq 3$.*

Proof. In order to verify the first inequality in (9), note that by the definitions of τ and C_0 in (12) and the obvious bound $x! < x^x$,

$$(13) \quad (k^2!)^2 \tau \leq k^{4k^2} 2^{4\sqrt{\log n}} R^{10/k} n^{-\frac{2}{k+1}}.$$

It remains to show that the R-H-S of (13) is smaller than one, or, by taking logarithms, that

$$4k^2 \log k + 4\sqrt{\log n} + \frac{10}{k} \log R < \frac{2}{k+1} \log n.$$

This, however, follows from

$$4\sqrt{\log n} < \frac{1}{k+1} \log n,$$

or equivalently,

$$16(k+1)^2 < \log n,$$

and from

$$4k^2(k+1) \log k + \frac{10}{k}(k+1) \log R < \log n,$$

both of which are true by the lower bound on n in (2).

To prove the second inequality in (9), note that since $\tau \leq 1$ and $j \leq \binom{l_j}{2}$, the quantity $\tau^{j-1} n^{l_j-2}$ is minimized when $j = \binom{l_j}{2}$. Thus, we have

$$(14) \quad \begin{aligned} \tau^{j-1} \cdot n^{l_j-2} &\geq \tau^{\binom{l_j}{2}-1} \cdot n^{l_j-2} = C_0^{\binom{l_j}{2}-1} n^{-\frac{(l_j-2)(l_j+1)}{k+1} + l_j - 2} \\ &= C_0^{\binom{l_j}{2}-1} n^{\frac{(l_j-2)(k-l_j)}{k+1}}. \end{aligned}$$

Recall that for $j \geq 2$ we have $l_j \geq 3$. In what follows we obtain a lower bound on the R-H-S of (14) by distinguishing two cases: $l_j < k$ and $l_j = k$. If $l_j < k$, then $(l_j - 2)(k - l_j)$ is minimized for $l_j = 3$ and $l_j = k - 1$ and owing to $C_0 > 1$ we infer

$$\tau^{j-1} \cdot n^{l_j-2} \stackrel{(14)}{\geq} C_0^{\binom{l_j}{2}-1} n^{\frac{(l_j-2)(k-l_j)}{k+1}} > n^{\frac{k-3}{k+1}} \stackrel{(2)}{\geq} k^{80k^4} R^{8k^2},$$

where we also used the bound $\frac{k+1}{k-3} \leq 5$ for all $k \geq 4$, which holds due to $3 \leq l_j < k$. If, on the other hand, $l_j = k$, then, by the definition of C_0 in (12) and the bound on n in (2),

$$(15) \quad C_0 \geq 2^{80k^2} R^{10/k}.$$

Hence, in view of (15), and the fact that $\binom{k}{2} - 1 \geq \frac{1}{5}k^2$ for $k \geq 3$, we have that

$$\tau^{j-1} \cdot n^{l_j-2} \stackrel{(14)}{\geq} C_0^{\binom{k}{2}-1} \geq \left(2^{80k^2} R^{10/k}\right)^{k^2/5} = 2^{16k^4} R^{2k}.$$

Consequently, using the trivial bounds $k^k \cdot k^{2!} < 2^{15k^4}$, $\binom{R}{k} < R^k$, and $R^k \stackrel{(1)}{>} 2r$, we conclude that

$$\sum_{j=2}^{\binom{k}{2}} \frac{2^{k^4} k^{k-2}}{\tau^{j-1} n^{l_j-2}} \leq \sum_{j=2}^{\binom{k}{2}} \frac{2^{k^4} k^{k-2}}{2^{16k^4} R^{2k}} \leq \frac{k^k}{2^{15k^4} R^{2k}} \leq \frac{1}{2r \binom{R}{k} \cdot k^{2!}} \stackrel{(10),(11)}{=} \frac{\varepsilon}{k^{2!}},$$

which concludes this proof. ■

In view of Claim 9, the conclusions of Corollary 7 hold true with ε and τ defined in, resp., (11) and (12). That is, there exists a collection \mathcal{C} of subgraphs of K_n such that Properties (a), (b), and (c) of Corollary 7 are satisfied for these specific values of ε and τ .

To continue with the proof of Lemma 4 consider a random graph $G(n, p)$ and let \mathcal{E} be the event that $G(n, p) \not\rightarrow (K_k)_r$. For each $G \in \mathcal{E}$, there exists an r -coloring $\varphi: E(G) \rightarrow [r]$ yielding no monochromatic copy of K_k . (Further on we will call such a coloring *proper*.) In other words, there are K_k -free graphs G_1, \dots, G_r , defined by $G_i = \varphi^{-1}(i)$, such that $G_1 \cup \dots \cup G_r = G$. According to Property (a) of Corollary 7, for every $i \in [r]$ there exists a graph $C_i \in \mathcal{C}$ such that $G_i \subseteq C_i$. Consequently,

$$G \cap \left(K_n \setminus \bigcup_{i=1}^r C_i \right) = \emptyset.$$

Notice that there are only at most $|\mathcal{C}|^r$ distinct graphs $K_n \setminus \bigcup_{i=1}^r C_i$. Moreover, we next show that all these graphs are dense (see Claim 10). Hence,

as it is extremely unlikely for a random graph $G(n, p)$ to be completely disjoint from one of the few given dense graphs, it will ultimately follow that $\mathbb{P}(\mathcal{E}) = o(1)$.

Claim 10. For all $C_1, \dots, C_r \in \mathcal{C}$,

$$\left| K_n \setminus \bigcup_{i=1}^r C_i \right| \geq \binom{n}{2} / R^2.$$

Proof. The graphs C_i , $i \in [r]$, together with $K_n \setminus \bigcup_{i=1}^r C_i$, form an $(r+1)$ -coloring of K_n , more precisely, an $(r+1)$ -coloring where, for each $i = 1, \dots, r$, the edges of color i are contained in C_i , while all edges of $K_n \setminus \bigcup_{i=1}^r C_i$ are colored with color $r+1$. (Note that this coloring may not be unique, as the graphs C_i are not necessarily mutually disjoint.) By Proposition 8, this $(r+1)$ -coloring yields either more than $(\alpha/2) \binom{n}{k}$ monochromatic copies of K_k in the first r colors or more than $\binom{n}{2} / R^2$ edges in the last color. Since for each $i \in [r]$, the i -th color class is contained in C_i , it follows from Property (b) that there are at most

$$r \cdot \varepsilon \binom{n}{k} \stackrel{(11)}{=} \frac{\alpha}{2} \binom{n}{k}$$

monochromatic copies of K_k in the first r colors. Consequently, we must have

$$(16) \quad \left| K_n \setminus \bigcup_{i=1}^r C_i \right| > \frac{1}{R^2} \binom{n}{2},$$

which concludes the proof. ■

Based on Claim 10 we can now bound $\mathbb{P}(\mathcal{E}) = \mathbb{P}(G(n, p) \not\rightarrow (K_s)_r)$ from above.

Claim 11.

$$\mathbb{P}(G(n, p) \not\rightarrow (K_s)_r) \leq |\mathcal{C}|^r \exp \left\{ -\frac{p \binom{n}{2}}{R^2} \right\}$$

Proof. Let \mathcal{F} be the event that $G(n, p) \cap (K_n \setminus \bigcup_{i=1}^r C_i) = \emptyset$ for at least one r -tuple of graphs $C_i \in \mathcal{C}$, $i = 1, \dots, r$. We have $\mathcal{E} \subseteq \mathcal{F}$. Indeed, if $G \in \mathcal{E}$ then there is a proper coloring φ of G and graphs $C_1, \dots, C_r \in \mathcal{C}$ such that $G \subseteq \bigcup_{i=1}^r C_i$ and, by Claim 10, $K_n \setminus \bigcup_{i=1}^r C_i$ has at least $\frac{1}{R^2} \binom{n}{2}$ edges and is disjoint from G . Thus, $G \in \mathcal{F}$. Consequently,

$$\mathbb{P}(G(n, p) \not\rightarrow (K_k)_r) \leq \mathbb{P}(\mathcal{F}).$$

To estimate $\mathbb{P}(\mathcal{F})$ we write $\mathcal{F} = \bigcup \mathcal{F}(C_1, \dots, C_r)$, where the summation runs over all collections (C_1, \dots, C_r) with $C_i \in \mathcal{C}$, $i = 1, \dots, r$, and the event $\mathcal{F}(C_1, \dots, C_r)$ means that $G(n, p) \cap (K_n \setminus \bigcup_{i=1}^r C_i) = \emptyset$. Clearly,

$$\mathbb{P}(\mathcal{F}(C_1, \dots, C_r)) = (1 - p)^{|K_n \setminus \bigcup_{i=1}^r C_i|} \leq (1 - p)^{\binom{n}{2}/R^2},$$

where the last inequality follows by Claim 10. Finally, applying the union bound, we have

$$\mathbb{P}(G(n, p) \not\rightarrow (K_s)_r) \leq \mathbb{P}(\mathcal{F}) \leq |\mathcal{C}|^r (1 - p)^{\binom{n}{2}/R^2} \leq |\mathcal{C}|^r \exp \left\{ -\frac{p \binom{n}{2}}{R^2} \right\}. \quad \blacksquare$$

Observe that by Property (c) of Corollary 7,

$$(17) \quad |\mathcal{C}|^r \leq \exp \left\{ r(2k^2)! \log(1/\varepsilon) \tau \log(1/\tau) \binom{n}{2} \right\}.$$

In view of Claim 11 and inequality (17), to complete the proof of Lemma 4, it suffices to show that

$$r(2k^2)! \log(1/\varepsilon) \tau \log(1/\tau) \binom{n}{2} \leq \frac{p \binom{n}{2}}{2R^2},$$

or, equivalently, after applying the definitions of p and τ ((3) and (12), resp.) and dividing sidewise by $n^{-\frac{2}{k+1}} \binom{n}{2}$, that

$$(18) \quad r(2k^2)! \log(1/\varepsilon) C_0 \log(1/\tau) \leq C/(2R^2).$$

To this end, observe that, since $C_0 \geq 1$ and, by (1), $R > 2r$, we have

$$\log(1/\tau) \stackrel{(12)}{\leq} \frac{2}{k+1} \log n$$

and

$$\log(1/\varepsilon) \stackrel{(11)}{=} \log(2r \binom{R}{k}) \leq (k + 1) \log R.$$

Hence, the L-H-S of (18) can be upper bounded by $2r(2k^2)! C_0 \log R \log n$. Consequently, using also the bounds $(2k^2)! < (2k)^{4k^2}$ and, again, $R > 2r$, we realize that (18) will follow from

$$(19) \quad 2R^3 \log R \cdot (2k)^{4k^2} \log n \leq C/C_0.$$

On the other hand,

$$C/C_0 \stackrel{(3),(12)}{=} 2^{5\sqrt{\log n} \log k - 4\sqrt{\log n}} R^{16-10/k} \geq 2^{\sqrt{\log n} \log k + 4\sqrt{\log n}(\sqrt{\log k} - 1)} R^{12}.$$

Thus, (19) is an immediate consequence of the following two inequalities, which are themselves easy consequences of (2):

$$2^{\sqrt{\log n \log k}} \stackrel{(2)}{\geq} 2^{20k^2 \log k} \geq (2k)^{4k^2}$$

and

$$2^{4\sqrt{\log n(\sqrt{\log k}-1)}} > 2^{\sqrt{\log n}} \geq \log n.$$

For the latter inequality we first used $k \geq 3$ and $\sqrt{\log 3} > \frac{5}{4}$, and then the fact that $2^{\sqrt{x}} \geq x$ for all $x \geq 16$, which can be easily verified by checking the first derivative (note that by (2), $\log n \geq 16$). This completes the proof of Lemma 4.

Remark 3. The idea of utilizing hypergraph containers for Ramsey properties of random graphs comes from a recent paper by Nenadov and Steger [15] (see also [11], Ch. 7) where the authors give a short proof of the main theorem from [20] establishing an edge probability threshold for the property $G(n, p) \rightarrow (F)_r$. Let us point to some similarities and differences between their and our approach. For clarity of the comparison, let us restrict ourselves to the case $F = K_k$ considered in our paper (the generalization to an arbitrary graph F is quite straightforward).

In [15] the goal is to prove an asymptotic result with $n \rightarrow \infty$ and all other parameters fixed. Consequently, they do not optimize, or even specify constants. Our task is to provide as good as possible upper bound on n in terms of k and r , so there is no asymptotics.

The observation that a K_k -free coloring of the edges of $G(n, p)$ yields r independent sets in the hypergraph $H(n, k)$, and therefore, by the Saxton-Thomason Theorem there are r graphs C_i , $i = 1, \dots, r$, each with only a few copies of K_k , whose union contains all the edges of $G(n, p)$, was made in [15]. Also there one can find a statement similar to our Proposition 8 (Corollary 3 in [15].) These two facts lead to similar estimates of the probability that $G(n, p)$ is not Ramsey. However, Nenadov and Steger, assuming that C is a constant, are forced to use Theorem 2.3 from [24] which involves the sequences of sets T_i . In our setting, we choose $C = C(n)$ in a balanced way, allowing us to go through with the estimates of $\mathbb{P}(G(n, p) \not\rightarrow (K_k)_r)$ without introducing the T_i 's, while, on the other hand, keeping the upper bound on n exponential in k . In fact, as observed by Conlon and Gowers [2], the approach via random graphs cannot yield a better than double-exponential upper bound on n if one assumes that p is at the Ramsey threshold, i.e., if C is a constant.

3. Relaxed Folkman numbers

In this section we prove Theorem 2. We will need an elementary fact about Ramsey properties of quasi-random graphs. For constants ϱ and d with $0 < d, \varrho \leq 1$, we say that an n -vertex graph Γ is (ϱ, d) -dense if every induced subgraph on $m \geq \varrho n$ vertices contains at least $d(m^2/2)$ edges. It follows by an easy averaging argument that it suffices to check the above inequality only for $m = \lceil \varrho n \rceil$. Note also that every induced subgraph of a (ϱ, d) -dense n -vertex graph on at least cn vertices is $(\frac{\varrho}{c}, d)$ -dense.

It turns out that for a suitable choice of the parameters, (ϱ, d) -dense graphs are Ramsey.

Proposition 12. *For every $k \geq 2$ and every $d \in (0, 1)$, if $n \geq (2/d)^{2k-4}$ and $0 < \varrho \leq (d/2)^{2k-4}$, then every two-colored n -vertex (ϱ, d) -dense graph Γ contains a monochromatic copy of K_k .*

Proof. For a two-coloring of the edges of a graph Γ we call a sequence of vertices (v_1, \dots, v_l) *canonical* if for each $i = 1, \dots, l-1$ all the edges $\{v_i, v_j\}$, for $j > i$, are of the same color.

We will first show by induction on l that for every $l \geq 2$ and $d \in (0, 1)$, if $n \geq (2/d)^{l-2}$ and $0 < \varrho \leq (d/2)^{l-2}$, then every two-colored n -vertex (ϱ, d) -dense graph Γ contains a canonical sequence of length l .

For $l=2$, every ordered pair of adjacent vertices is a canonical sequence. Assume that the statement is true for some $l \geq 2$ and consider an n -vertex (ϱ, d) -dense graph Γ , where $\varrho \leq (d/2)^{l-1}$ and $n \geq (2/d)^{l-1}$. As observed above, there is a vertex u with degree at least dn . Let M_u be a set of at least $dn/2$ neighbors of u connected to u by edges of the same color. Let $\Gamma_u = \Gamma[M_u]$ be the subgraph of Γ induced by the set M_u . Note that Γ_u has $n_u \geq dn/2 \geq (2/d)^{l-2}$ vertices and is (ϱ_u, d) -dense with $\varrho_u \leq (d/2)^{l-2}$. Hence, by the induction assumption, there is a canonical sequence of length l in Γ_u . This sequences preceded by the vertex u makes a canonical sequence of length $l+1$ in Γ .

To complete the proof of Proposition 12, set $l=2k-2$ above and observe that every canonical sequence (v_1, \dots, v_{2k-2}) contains a monochromatic copy of K_k . Indeed, among the vertices v_1, \dots, v_{2k-3} , some $k-1$ have the same color on all the “forward” edges. These vertices together with vertex v_{2k-2} form a monochromatic copy of K_k . ■

Proof of Theorem 2. Let $n = 2^{4k/(1-4\alpha)}$. Consider a random graph $G(n, p)$ where

$$p = 2n^{-\frac{7+4\alpha}{16k}} = 2^{-\frac{20\alpha+3}{4(1-4\alpha)}}.$$

By elementary estimates one can bound the expected number of l -cliques in $G(n, p)$ by

$$\left(\frac{en}{l} p^{\frac{l-1}{2}}\right)^l.$$

Thus, if

$$\frac{l-1}{2} \geq \frac{\log n}{\log(1/p)} = \frac{16k}{20\alpha + 3},$$

then, as $k \rightarrow \infty$, a.a.s. there are no l -cliques in $G(n, p)$. By assumption,

$$\frac{l-1}{2} \geq \frac{k-\alpha}{2\alpha} \geq \frac{16k}{20\alpha + 3},$$

where the last inequality, equivalent to $(3 - 12\alpha)k \geq 20\alpha^2 + 3\alpha$, holds if $k \geq \frac{2}{3(1-4\alpha)}$ (we used here the assumption that $\alpha < \frac{1}{4}$).

Further, by a straightforward application of Chernoff’s bound (see, e.g., [13, ineq. (2.6)]), a.a.s. $G(n, p)$ is $(\varrho, p - o(p))$ -dense, where $\varrho = \frac{\log^2 n}{n}$, say. Indeed, setting $t = \varrho n = \log^2 n$, $\epsilon = \epsilon(n) = (\log n)^{-1/3}$, and $d = (1 - \epsilon)p$, the probability that a fixed set T of t vertices spans in $G(n, p)$ fewer than $dt^2/2$ edges is at most

$$\begin{aligned} \mathbb{P}(e(T) \leq (1 - \epsilon)pt^2/2) &\leq \mathbb{P}\left(e(T) \leq (1 - \epsilon/2)p \binom{t}{2}\right) \\ &\leq \exp\left\{-\frac{\epsilon^2}{8}p \binom{t}{2}\right\} \leq \exp\left\{-\frac{\epsilon^2}{24}pt^2\right\}. \end{aligned}$$

Finally, note that the above bound, even multiplied by $\binom{n}{t}$, the number of all t -element subsets of vertices in $G(n, p)$, still converges to zero (recall that p is a constant).

Using that $\epsilon k = O(\log^{2/3} n)$ one can easily verify that both assumptions of Proposition 12, that is, $n \geq (2/d)^{2k-4}$ and $\varrho \leq (d/2)^{2k-4}$, hold true. Indeed, dropping the subtrahend 4 for simplicity,

$$(d/2)^{2k} = (1 - \epsilon)^{2k} n^{-1+\delta} \geq \varrho \geq \frac{1}{n},$$

for n large enough, that is, for k large enough.

In conclusion, a.a.s. $G(n, p)$ is such that

- it contains no K_l , and
- for every two-coloring of its edges, there is a monochromatic copy of K_k .

Hence, there exists an n -vertex graph with the above two properties and, consequently, $f(k, l) \leq n = 2^{4k/(1-4\alpha)}$. ■

4. Hypergraph Folkman numbers

Hypergraph Folkman numbers are defined in an analogous way to their graph counterparts. Given three integers h , k , and r , the h -uniform Folkman number $f_h(k; r)$ is the minimum number of vertices in an h -uniform hypergraph H such that $H \rightarrow (K_k^{(h)})_r$ but $H \not\rightarrow K_{k+1}^{(h)}$. Here $K_k^{(h)}$ stands for the complete h -uniform hypergraph on k vertices, that is, one with $\binom{k}{h}$ edges. The finiteness of hypergraph Folkman numbers was proved by Nešetřil and Rödl in [17, Colloary 6, page 206] and besides the gigantic upper bound stemming from their construction, no reasonable bounds have been proven so far. Much better understood are the vertex-Folkman numbers (where instead of edges, the vertices are colored), which for both, graphs and hypergraphs, are bounded from above by an almost quadratic function of k , while from below the bound is only linear in k (see [6,4]).

The study of Ramsey properties of random hypergraphs began in [21] where a threshold was found for $K_4^{(3)}$, the 3-uniform clique on 4 vertices. Also there a general conjecture was stated that a theorem analogous to that in [20] holds for hypergraphs too. This was confirmed for h -partite h -uniform hypergraphs in [22], and, finally, for all h -uniform hypergraphs in [9] and, independently, in [3].

As remarked by Nenadov and Steger in [15], the Saxton-Thomason (or the Balogh-Morris-Samotij) theorem should also yield a much simpler proof of the hypergraph Ramsey threshold theorem from [9,3]. We believe that, similarly, our quantitative approach should also provide an upper bound on the hypergraph Folkman numbers $f_h(k; r)$, exponential in a polynomial of k and r .

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Vojtěch Rödl

Emory University

Atlanta, USA

rod1@mathcs.emory.edu

Andrzej Ruciński

A. Mickiewicz University

Poznań, Poland

rucinski@amu.edu.pl

Mathias Schacht

Universität Hamburg

Hamburg, Germany

schacht@math.uni-hamburg.de