# ON MINIMAL VERTEX - FOLKMAN GRAPHS

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ABSTRACT. Following the arrow notation, for a graph $G$ and natural numbers $a_1, a_2, \ldots, a_r$ we write $G \to (a_1, a_2, \ldots, a_r)^v$ if for every coloring of the vertices of $G$ with $r$ colors there exists a copy of the complete graph $K_{a_i}$ of color $i$ for some $i = 1, 2, \ldots, r$. We present some constructions of small graphs with this Ramsey property, but not containing large cliques. We also set bounds on the order of the smallest such graphs.

## 1. Introduction

Let $G$ be a graph and let $a_1, a_2, \ldots, a_r$ be positive integers. We write $G \to (a_1, a_2, \ldots, a_r)^v$ if every $r$-coloring of the vertices of $G$ forces a complete subgraph $K_{a_i}$ of color $i$ for some $i \in \{1, 2, \ldots, r\}$. It is a simple consequence of the pigeon-hole principle that $K_m$, where

$$m = m(a_1, a_2, \ldots, a_r) = 1 + \sum_{i=1}^{r} (a_i - 1)$$

is the minimal graph $G$ such that $G \to (a_1, a_2, \ldots, a_r)^v$. Obviously, every graph containing $K_m$ satisfies the above property too. A natural question arises what other graphs satisfy this property. For a positive integer $w > \max\{a_1, a_2, \ldots, a_r\}$ set

$$\mathcal{G}^v(a_1, a_2, \ldots, a_r; w) = \{G : G \to (a_1, a_2, \ldots, a_r)^v \text{ and } K_w \not\subseteq G\}.$$

Note that if $a_i = 1$ for some $i \in \{1, 2, \ldots, r\}$, then

$$G \to (a_1, a_2, \ldots, a_r)^v \iff G \to (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r)^v.$$

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Hence we will be assuming that $a_i \geq 2$ for $i \in \{1, 2, \ldots, r\}$. Observe also that if $\sigma$ is a permutation of the set $\{1, 2, \ldots, r\}$, then

$$G \to (a_1, a_2, \ldots, a_r)^v \iff G \to (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(r)})^v.$$

Thus, we may and shall assume that $a_1 \leq a_2 \leq \cdots \leq a_r$.

As mentioned above, $K_m \in \mathcal{G}^v(a_1, a_2, \ldots, a_r; w)$ for $w > m$. In 1970 Folkman [3] proved that for every sequence $a_1, a_2, \ldots, a_r$ and for every $w > a_r$, the family $\mathcal{G}^v(a_1, a_2, \ldots, a_r; w)$ is nonempty. The elements of this family will be called the *vertex-Folkman graphs.*

One can ask what is the minimum number of vertices in a graph from the family $\mathcal{G}^v(a_1, a_2, \ldots, a_r; w)$. Let us define

$$F^v(a_1, a_2, \ldots, a_r; w) = \min\{|V(G)| : G \in \mathcal{G}^v(a_1, a_2, \ldots, a_r; w)\}.$$

These numbers are called the *vertex-Folkman numbers.* Except for the last section, we will refer to them as to Folkman numbers. A modification of Folkman's construction yielding a recursive bound on Folkman numbers is given in Section 5.

We now present a simple lower bound on the Folkman numbers in general case.

**Proposition 1.** *Let* $m = 1 + \sum_{i=1}^{r}(a_i - 1)$ *and let* $m < w < a_r$. *Then*

$$F^v(a_1, a_2, \ldots, a_r; w) \geq 2m - w + 1 \ .$$

*Proof.*

Let $G$ be a graph with $2m - w$ vertices which contains no clique of order $w$. Then the complement $G^c$ of the graph $G$ contains a matching $M$ of size at least $m - w + 1$. (In fact, *every* maximal matching of $G^c$ has at least $m - w + 1$ edges.) Since

$$|V(G)| - |M| \leq m - 1 = \sum_{i=1}^{r}(a_i - 1) \ ,$$

one can color the set $V(G)$ using only $r$ colors, and so that, for $i = 1, \ldots, r$, no clique of order $a_i$ appears in the $i$-th color. (Just color both ends of each edge of $M$ by the same color.) $\square$

The lower bounds obtained in this paper are often better than the bound in Proposition 1. In the forthcoming sections we will be gradually increasing the level of difficulty by taking $w = m - k$ for $k$ bigger and bigger. In the first nontrivial case, $k = 0$, we determine precisely the Folkman numbers and the unique graphs yielding them. In Section 3 the next case $k = 1$ is studied. The obtained bounds are quite tight, but the exact value of the Folkman number is known only in the smallest case $a_r = 2$. Section 4 contains some constructive and nonconstructive results bounding the Folkman numbers for not too large $k$. In Section 5 we consider the most difficult case when $w = a_r + 1$, the smallest possible value. Here the bounds are not satisfactory. Finally, the last section links the vertex-Folkman graphs with the edge Folkman graphs. We show how upper bounds in the vertex case imply corresponding bounds in the edge case. The Appendix contains the proof of a graph theoretic result (Lemma 1) needed i n the proof of Theorem 3 from Section 2.

## 2. Forbidden clique: $K_m$

In this section we consider the case of the greatest $w$ for which the problem is nontrivial, i.e. the Folkman numbers of the form $F^v(a_1, a_2, \ldots, a_r; w)$, where $w = m = 1 + \sum_{i=1}^{r}(a_i - 1)$. Note that in this case Proposition 1 yields only a trivial lower bound. We will first study the minimality of Folkman graphs with respect to deleting vertices and edges. In the second part we will find the unique smallest Folkman graph.

### 2.1. Minimal Folkman graphs

In the following we will need some standard graph theoretic notation. The *join* $H + G$ of two vertex-disjoint graphs $H$ and $G$ is the graph with vertex set $V(H) \cup V(G)$ and edge set $E(H) \cup E(G) \cup \{\{u, v\} : v \in V(H), u \in V(G)\}$. For a connected graph $G$ and for two of its vertices $u, v$, let the *distance* $d_G(u, v)$ between the vertices be the length of a shortest path in $G$ with the endpoints in $u$ and $v$. We will say that a graph $H$ is the *n-th power* of a graph $G$ and write $H = G^n$, if $V(H) = V(G)$ and $E(H) = \{\{u, v\} : d_G(u, v) \leq n\}$. By $C_n$ we will denote a cycle of length $n$. For a graph $H$ and for an integer $n \geq |V(H)|$, let $K_n - H$ stand for the graph obtained from the complete graph $K_n$ by deleting the edges of a copy of the graph $H$. The union of $n$ vertex disjoint copies of $H$ will be designated by $nH$.

The following theorem shows a large class of vertex-Folkman graphs.

**Theorem 1.** *Let $k \in \{a_r, a_r + 1, \ldots, m - 1\}$ and let $n$ be an integer greater than $2k$. Denote by $s$ the greatest common divisor of $n$ and $k$ and put $t = k/s$. Furthermore, let $G = C_n^{k-1} + K_{m-k-1}$.*

*Then $G \in \mathcal{G}^v(a_1, a_2, \ldots, a_r; m)$ if and only if $\sum_{i=1}^{r} \lfloor \frac{a_i - 1}{t} \rfloor < s$.*

*Proof.*

Let $G = C_n^{k-1} + K_{m-k-1}$. It is easy to note that $G$ contains no clique of order $m$. Let $u_1, u_2, \ldots, u_{m-k-1}$ be the vertices in $K_{m-k-1}$ and let $v_0, v_1, \ldots, v_{n-1}$ be the vertices in $C_n^{k-1}$. Assume also that this labelling has the property that for $j = 0, 1, \ldots, n-2$ we have $\{v_j, v_{j+1}\} \in E(C_n)$ and $\{v_0, v_{n-1}\} \in E(C_n)$. In order to prove the first of the implications, suppose to the contrary that $\sum_{i=1}^{r} \lfloor \frac{a_i - 1}{t} \rfloor < s$ and $G \notin \mathcal{G}^v(a_1, a_2, \ldots, a_r; m)$. Let $h : V(G) \to \{1, 2, \ldots, r\}$ be such a coloring of the vertices of $G$ that there is no $K_{a_i}$ of color $i$ in $G$. Consider the subgraph induced by the vertices $\{u_1, u_2, \ldots, u_{m-k-1}, v_0, v_1, \ldots, v_{k-1}\}$. It is isomorphic to $K_{m-1}$, so for every $i \in \{1, 2, \ldots, r\}$ it has precisely $a_i - 1$ vertices of color $i$. There is the same situation in the subgraph induced by the vertices $\{u_1, u_2, \ldots, u_{m-k-1}, v_1, \ldots, v_k\}$. This implies that $h(v_0) = h(v_k)$ and consequently if

$p+k \equiv q(\text{mod } n)$ then $h(v_p) = h(v_q)$. Note that since $s$ is the greatest common divisor of $n$ and $k$, we have also $h(v_p) = h(v_q)$ if $p+s \equiv q(\text{mod } n)$. Thus the colors of the vertices $v_0, v_1, \ldots, v_{s-1}$ determine the colors of the whole cycle. Recall that the vertices $v_0, v_1, \ldots, v_{k-1}$ induce a clique, so at most $a_i - 1$ of them have color $i$ (for $i = 1, 2, \ldots, r$), hence at most $\lfloor \frac{a_i-1}{t} \rfloor$ of the vertices $v_0, v_1, \ldots, v_{s-1}$ have color $i$. Let $b_i$ be the number of vertices of color $i$ in $\{v_0, v_1, \ldots, v_{s-1}\}$. We have $s = \sum_{i=1}^{r} b_i \leq \sum_{i=1}^{r} \lfloor \frac{a_i-1}{t} \rfloor < s$, which is a contradiction.

Now suppose that $\sum_{i=1}^{r} \lfloor \frac{a_i-1}{t} \rfloor \geq s$. We will find a coloring $h : V(G) \to \{1, 2, \ldots, r\}$ such that $G$ contains no $K_{a_i}$ of color $i$, $i = 1, 2, \ldots, r$. We begin with choosing the colors for $\{v_0, v_1, \ldots, v_{s-1}\}$. From our assumption we can color less than $\frac{a_i}{t}$ of them by the $i$th color, $i = 1, 2, \ldots, r$. Take any such coloring and color every other vertex of the cycle $C_n$ in such a way that $h(v_p) = h(v_q)$ if $p+s \equiv q(\text{mod } n)$. Let $b_i$ be the number of vertices of color $i$ in $\{v_0, v_1, \ldots, v_{s-1}\}$, $i = 1, 2, \ldots, r$. Note that every maximal clique induced in $G$ by a subset of the vertices $v_0, v_1, \ldots, v_{n-1}$ consists precisely of $k$ consecutive vertices of $C_n$. In our coloring every such $K_k$ has exactly $tb_i$ vertices of color $i$, $i = 1, 2, \ldots, r$. Now color all the vertices $u_1, u_2, \ldots, u_{m-k-1}$ using the $i$th color at most $a_i - tb_i - 1$ times, $i = 1, 2, \ldots, r$. It is always possible since $m-k-1 = \sum_{i=1}^{r}(a_i - 1) - k \leq \sum_{i=1}^{r}(a_i-1) - \sum_{i=1}^{r} tb_i = \sum_{i=1}^{r}(a_i - tb_i - 1)$. Note that every maximal clique in $G$ contains at most $(tb_i) + (a_i - tb_i - 1) = a_i - 1$ vertices of color $i$, $i = 1, 2, \ldots, r$. $\square$

**Corollary 1.** *If $k$ and $n$ are relatively prime, then $C_n^{k-1} + K_{m-k-1} \in \mathcal{G}^v(a_1, a_2, \ldots, a_r; m)$.*

*Proof.*

It is enough to note that if $k$ and $n$ are relatively prime then we have $s = 1$ and $t = k$, and thus $\sum_{i=1}^{r} \lfloor \frac{a_i-1}{t} \rfloor = 0 < s = 1$. $\square$

In the following we will consider the graphs $G = G(k, n)$ from Theorem 1 with $k = a_r$ and $n \geq 2k+1$ such that $k$ and $n$ are relatively prime. Note that such graphs create an infinite class of vertex-Folkman graphs. Are all these graphs minimal?

A graph $H \in \mathcal{G}^v(a_1, a_2, \ldots, a_r; m)$ is called $(a_1, a_2, \ldots a_r)$ - *vertex minimal* (or simply *vertex minimal*) if for every vertex $v \in V(H)$ we have $H - v \notin \mathcal{G}^v(a_1, a_2, \ldots, a_r; m)$. Note that a graph $H \in \mathcal{G}^v(a_1, a_2, \ldots, a_r; m)$ is vertex minimal if and only if it is vertex-Ramsey minimal, i.e. if $H - v \not\to (a_1, a_2, \ldots a_r)^v$ for every vertex $v \in V(H)$.

A graph $H \in \mathcal{G}^v(a_1, a_2, \ldots, a_r; m)$ is called $(a_1, a_2, \ldots a_r)$ - *edge minimal* (or simply *edge minimal*) if for every edge $e \in E(H)$ we have $H - e \notin \mathcal{G}^v(a_1, a_2, \ldots, a_r; m)$. Obviously, $H$ is edge minimal if and only if it is edge-Ramsey minimal, i.e. if $H - e \not\to (a_1, a_2, \ldots a_r)^v$ for every edge $e \in E(H)$.

We first consider the vertex minimality.

**Proposition 2.** *Let $k = a_r$, $n \geq 2k+1$ and assume that $k$ and $n$ are relatively prime. Then the graph $C_n^{k-1} + K_{m-k-1}$ is $(a_1, a_2, \ldots, a_r)$-vertex minimal.*

*Proof.*

As in the proof of Theorem 1 we denote the vertices of the clique $K = K_{m-k-1}$ by $u_0, u_1, \ldots, u_{m-k-1}$ and the vertices of the cycle $C_n$ by $v_0, v_1, \ldots, v_{n-1}$. Assume also that this labelling has the property that for $j = 0, 1, \ldots, n-2$ we have $\{v_j, v_{j+1}\} \in E(C_n)$ and $\{v_0, v_{n-1}\} \in E(C_n)$. To prove Proposition 2 we remove a vertex from $G$ and show that the obtained graph $G'$ does not satisfy the Ramsey property $G' \to (a_1, a_2, \ldots a_r)^v$.

First assume that the removed vertex belongs to the cycle $C_n$. By symmetry we may assume that it is $v_0$. Assign color 1 to the vertices $v_{a_r}, v_{2a_r}, \ldots v_{\lfloor \frac{n}{a_r} \rfloor a_r}$ and color $r$ to all the remaining vertices of the cycle $C_n$. Obviously, there is no $K_2$ colored 1 and no $K_{a_r}$ colored $r$. Now assign color 1 to $a_1 - 2$ arbitrary vertices of the clique $K$, and, for every $i = 2, 3, \ldots, r-1$, assign color $i$ to exactly $a_i - 1$ vertices of $K$. Of course , there is no $K_{a_i}$ of color $i$ for every $i = 1, 2, \ldots, r$ and every vertex in $G' = G - v_0$ has a color assigned, since

$$a_1 - 2 + \sum_{i=2}^{r-1}(a_i - 1) = \sum_{i=1}^{r-1}(a_i - 1) - 1 = m - a_r - 1.$$

Now assume that we remove a vertex $u$ from the clique $K_{m-k-1}$. We must then have $m - a_r - 1 \geq 1$, and either $r \geq 3$ or $a_1 \geq 3$ by the definition of $m$. Consider two cases:

*Case 1.* $a_1 = 2$.

Then $r \geq 3$. Assign color 1 to the vertices $v_0, v_k, v_{2k}, \ldots, v_{(\lfloor n/k \rfloor - 1)k}$ and color 2 to the vertex $v_{\lfloor n/k \rfloor k}$. All the remaining vertices on the cycle color by the $r$-th color. One can easily note that there is no $K_{a_r}$ of color $r$ and no $K_2$ with both endpoints colored by 1 or both by 2. In the clique $K$ we assign color 2 to $a_2 - 2$ vertices and, for every $i = 3, 4, \ldots, r-1$, we assign color $i$ to $a_i - 1$ vertices. There is no $K_{a_i}$ of color $i$ for $i = 1, 2, \ldots, r$ and every vertex in $G' = G - u$ has a color assigned, since

$$a_2 - 2 + \sum_{i=3}^{r-1}(a_i - 1) = \sum_{i=2}^{r-1}(a_i - 1) - 1 = \sum_{i=1}^{r-1}(a_i - 1) - 2 = m - a_r - 2\ .$$

*Case 2.* $a_1 > 2$.

In this case assign color 1 to the vertices $v_0, v_k, \ldots, v_{\lfloor \frac{n}{k} \rfloor k}$ and color $r$ to all the remaining vertices of the cycle. There is no $K_2$ colored 1, nor $K_{a_r}$ colored $r$. In the clique $K$ assign color 1 to $a_1 - 3$ vertices and for every $i = 2, 3, \ldots, r-1$ assign color $i$ to $a_i - 1$ vertices. We have

$$a_1 - 3 + \sum_{i=2}^{r-1}(a_i - 1) = m - a_r - 2$$

and thus there is no $K_{a_i}$ of color $i$, for every $i = 1, 2, \ldots, r$. □

It follows that every graph $G = C_n^{a_r - 1} + K_{m-a_r-1}$ with $a_r$ and $n$ relatively prime, contains an edge minimal subgraph of order $|V(G)|$, but not everyone is edge minimal itself. For example, an inquiring reader can check that the graph $C_{11}^3 + K_1$ is not $(2, 2, 4)$-edge minimal (remove the edge $\{v_0, v_2\}$). However, the following Proposition 3 shows explicitly an infinite class of graphs of the form $G = C_n^{a_r - 1} + K_{m-a_r-1}$ which are $(a_1, a_2, \ldots, a_r)$-edge minimal. We will use this Proposition 3 in the proof of Theorem 3.

**Proposition 3.** *If $n \geq 2a_r+1$ and $n \equiv 1(mod\ a_r)$ then $C_n^{a_r-1}+K_{m-a_r-1}$ is $(a_1, a_2, \dots, a_r)$-edge minimal.*

*Proof.*

Note that if $n \equiv 1(\text{mod } a_r)$ then $a_r$ and $n$ are relatively prime, and hence $C_n^{a_r-1} + K_{m-a_r-1} \to (a_1, a_2, \dots, a_r)^v$ by Corollary 1. Remove any edge from the graph $G = C_n^{a_r-1}+K_{m-a_r-1}$. We will show a coloring of the vertices of the obtained graph $G-e$ which certifies that $G - e \not\to (a_1, a_2, \dots, a_r)^v$. First assume that the removed edge $e = \{u, v\}$ has the property that $N(v) \subset N(u)$. Let us color all the vertices in such a way that the vertices $u, v$ have the same color and that the graph $G - v$ (which is a subgraph of $G - e$) contains no clique $K_{a_i}$ of color $i$ for every $i \in \{1, 2, \dots, r\}$. The existence of such a coloring follows from Proposition 2. One can check that there is no such a clique in the whole graph $G - e$. Indeed, suppose to the contrary that $v$ belongs to a clique $K_{a_j}$ with all vertices of color $j$. Denote the vertices of this clique by $x_1, x_2, \dots, x_{a_j-1}, x_{a_j} = v$. Note that $\{x_1, x_2, \dots, x_{a_j-1}\} \subset N(v) \subset N(u)$ and since $u$ has the $j$th color, the vertices $x_1, x_2, \dots, x_{a_j-1}, u$ create the clique with $a_j$ vertices of the $j$th color, contained in $G - v$ - a contradiction.

Note that if the removed edge has at least one endpoint (say $u$) in the clique $K_{m-a_r-1}$ then $N(v) \subset N(u)$. Thus, in the remaining case, we may assume that $u, v \in C_n^{a_r-1}$. Assume also, without loss of generality, that $e = \{v_0, v_p\}$, $p \leq a_r - 1$. Assign color 1 to the vertices $v_{a_r}, v_{2a_r}, \dots, v_{n-1}$ (there exists an integer $l$ such that $la_r = n - 1$ since $n \equiv 1(\text{mod } a_r)$ ). Obviously there is no edge with both endpoints of color 1. Now assign color $r$ to all remaining vertices in $C_n^{a_r-1}$. Note that $C_n^{a_r-1}$ contains no clique $K_{a_r}$ with all vertices of color $r$. Indeed, every clique of order $a_r$ in $C_n^{a_r-1}$ consists of $a_r$ consecutive vertices of the cycle $C_n$. The only set of $a_r$ consecutive vertices colored by the $r$-th color is $\{v_0, v_1, \dots, v_{a_r-1}\}$, but it does not induce a clique - we removed the edge $e = \{v_0, v_p\}$, $p \leq a_r - 1$. Now we can color the vertices in $K_{m-a_r-1}$. Assign color 1 to $a_1 - 2$ vertices and for every $i = 2, 3, \dots, r-1$, assign the $i$-th color on $a_i - 1$ vertices. Obviously, for every $i = 1, 2, \dots, r$, there is no $K_{a_i}$ of color $i$ in $G - e$. □

## 2.2. The smallest Folkman graph

Using Theorem 1 one can bound some Folkman numbers.

**Corollary 2.** $F^v(a_1, a_2, \dots, a_r; m) \leq a_r + m$

*Proof.*

Assume $k = a_r$ and $n = 2a_r + 1$. (Note that $k$ and $n$ are relatively prime.) The graph $C_n^{k-1}+K_{m-k-1}$ has $a_r+m$ vertices and, by Corollary 1, belongs to $\mathcal{G}^v(a_1, a_2, \dots, a_r; m)$. □

Recently it has been proved that the inequality in Corollary 2 cannot be improved any further.

**Theorem 2.** *([7])*

*For every non-descending sequence of natural numbers $a_1, a_2, \dots, a_r$ and $m = \sum_{i=1}^{r}(a_i - 1) + 1$ we have*

$$F^v(a_1, a_2, \dots, a_r; m) = a_r + m. \quad \square$$

To prove the upper bound of this statement the authors considered the graph $K_{a_r+m} - C_{2a_r+1}$ which is isomorphic to $C_{2a_r+1}^{a_r-1} + K_{m-a_r-1}$ used in Corollary 2. The main result of this section states that this is the unique smallest graph in $\mathcal{G}^v(a_1, a_2, \dots, a_r; m)$.

**Theorem 3.** *The graph $K_{a_r+m} - C_{2a_r+1}$ is the unique $(a_r + m)$-vertex graph $G$ with properties $G \to (a_1, a_2, \dots, a_r)^v$ and $K_m \not\subset G$.*

The main idea of the proof is taken from [7]. We exclude all $(a_r + m)$-vertex graphs except for $K_{a_r+m} - C_{2a_r+1}$. To do this we will need a graph-theoretic lemma concerning stable graphs. This fact, of some independent interest, will be proved in the appendix.

A graph $G$ is called *stable* if for every vertex $w \in V(G)$ and for every pair of adjacent vertices $\{u, v\} \in E(G)$ we have $\alpha(G) = \alpha(G - w) = \alpha(G')$, where $G'$ is the graph obtained from $G$ after deleting both vertices $u$ and $v$ and $\alpha(G)$ stands for the independence number of $G$, i.e the order of a largest independent set in $G$.

**Lemma 1.** *Let $G$ be a stable graph of order $n$. Then*

(i) $\alpha(G) < n/2$
(ii) *if $n = 2k + 1$ and $\alpha(G) = k$ then $G$ is isomorphic to the cycle $C_n$*

*Proof of Theorem 3.*

Let $G$ be a graph of order $a_r + m$ not containing $K_m$ and not isomorphic to $K_{a_r+m} - C_{2a_r+1}$. We shall consider the complement $G^c$ of $G$, and show that there exists a coloring of $G^c$ such that for $i = 1, 2, \dots, r$, no independent set of size $a_i$ is colored with the $i$-th color.

Let $S$ be a largest independent set in $G^c$. Since $K_m \not\subseteq G$ we have $|S| \le m - 1$ and thus $|V(G^c) \setminus S| \ge a_r + 1$. Denote by $T$ a subset of vertices of $G^c$ such that $T \cap S = \emptyset$ and $|T| = a_r + 1$, and let $H$ be the bipartite subgraph of $G^c$ induced by all edges with one end in $T$ and the other in $S$. We consider two cases.

*Case 1.* $H$ contains a matching $M$ which saturates all vertices of $T$.

Let $e_1, e_2$ be any edges from $M$. Color the vertices of $G^c$ in the following way:

(i) $2a_r - 2$ vertices incident to $M - \{e_1, e_2\}$ are colored with the $r$-th color;
(ii) all endpoints of the edges $e_1, e_2$, as well as $a_{r-1} - 3$ other vertices, are colored with the $(r - 1)$-th color;
(iii) among the remaining $a_r + m - (2a_r - 2) - (a_{r-1} + 1) = \sum_{i=1}^{r-2} (a_i - 1)$ vertices, precisely $a_i - 1$ are colored with the $i$-th color, for $i = 1, 2, \dots, r - 2$.

It is easy to see that in the above coloring the $i$-th color class contains no independent set of $G^c$ of order $a_i$, which completes the proof of this case.

*Case 2.* $H$ contains no matching saturating all vertices of $T$.

Let $T' \subseteq T$ be a maximal subset of $T$ such that its neighborhood in $H$ is smaller than the set itself, i.e. $|N_H(T')| < |T'|$. From Hall's theorem it follows that such a set $T'$ exists and furthermore, that $H$ contains a matching $M'$ (possibly empty) which saturates all vertices from $T \setminus T'$ and none from $N_H(T')$. Denote by $U$ the set of all vertices from $S$ saturated by $M'$, and put $W = T \cup N_H(T') \cup U$.

Observe that the subgraph $G^c(W)$ induced by $W$ in $G^c$ contains no independent sets of size larger than $|N_H(T')| + |U|$. Indeed, otherwise for such a set $I$ we have

$$|I \cap (T' \cup N_H(T'))| > |N_H(T')| ,$$

and the set $S \cup (I \cap T') \setminus (N_H(T') \setminus I)$ is an independent set of $G^c$ with

$$|S| + |I \cap T'| - |N_H(T')| + |I \cap N_H(T')| = |S| + |I \cap (T' \cup N_H(T'))| - |N_H(T')| > |S|$$

vertices, contradicting our choice of $S$. Thus, the independence number of $G^c(W)$ is bounded from above by $|N_H(T')| + |U|$ and consequently, if we add to $W$ any $2a_r + 1 - |W|$ vertices, the subgraph $W'[G^c]$ of $G^c$ induced by the resulting set $W'$ has $2a_r + 1$ vertices and no independent set of size $a_r + 1$. Now we consider two cases with respect to whether the subgraph of $G^c$ induced by $W'$ is or is not a stable graph.

*Case 1.* $W'[G^c]$ is not stable.
Thus we can find a vertex or an edge in $W'$, the deletion of which decreases the independence number of $W'$. If there exists a vertex $v \in W'$ such that $\alpha(W' \setminus \{v\}) \leq a_r - 1$ then color $W' \setminus \{v\}$ with the $r$-th color, and the vertex $v$, as well as $a_{r-1} - 2$ other vertices with the $(r-1)$-th color. If there is no such a vertex $v$, then there exists an edge $e = \{v_1, v_2\} \in W'[G^c]$, such that $\alpha(W' \setminus \{v_1, v_2\}) \leq a_r - 1$, since $W'[G^c]$ is not stable. Then color $W' \setminus \{v_1, v_2\}$ with the $r$-th color, both $v_1, v_2$ as well as $a_{r-1} - 2$ other vertices color with the $(r-1)$-th color. Now one can color the remaining $\sum_{i=1}^{r-2}(a_i - 1)$ vertices in such a way that for $i = 1, 2, \ldots, r-1$ the $i$-th color is used $a_i - 1$ times.

*Case 2.* $W'[G^c]$ is stable.
¿From Lemma 1 $W'[G^c]$ is isomorphic to a cycle $C_{2a_r+1}$ and therefore $G$ is a proper subgraph of $K_{a_r+m} - C_{2a_r+1}$. Proposition 3 implies that $K_{a_r+m} - C_{2a_r+1}$ is $(a_1, a_2, \ldots, a_r)$-edge minimal, thus $G \not\to (a_1, a_2, \ldots, a_r)^v$ and the proof is completed. □

## 3. Forbidden clique: $K_{m-1}$

In this section we go one step further and forbid cliques of size $m - 1$. Under this constrain it is much harder to derive such precise results like in the previous case of forbidding $K_m$. Therefore, most of our results here are just bounds on the Folkman numbers whose exact determination is still to come. Note that Propositi on 1 says only that $F^v(a_1, a_2, \ldots, a_r; m-1) \geq m + 2$. A better lower bound follows from Theorems 2 and 3.

**Corollary 3.**

$$F^v(a_1, a_2, \ldots, a_r; m-1) \geq a_r + m + 1$$

*Proof.*

By Theorem 3

$$F^v(a_1, a_2, \ldots, a_r; m) < F^v(a_1, a_2, \ldots, a_r; m-1) . \quad \square$$

As for the upper bound, we provide a quite general result, where we only impose that the largest integer $a_r$ is not too large when compared to the sum of the others.

**Proposition 4.** *If $m \geq 2a_r + 2$ then*

$$F^v(a_1, a_2, \ldots, a_r; m-1) \leq a_r^2 + m$$

**Remark:** If $a_r + 3 \leq m \leq 2a_r + 1$ then a little modification of the same construction gives an ugly bound $F^v(a_1, a_2, \ldots, a_r; m-1) \leq 3a_r^2 + a_r - ma_r + 2m - 3$ . When $m = a_r + 2$ then the forbidden clique has the smallest possible size of $a_r + 1$. This case is studied in Section 5 in more generality.

*Proof.*

We construct a small graph $G$ in $\mathcal{G}^v(a_1, a_2, \ldots, a_r; m-1)$ using the smallest graph $H = K_{a_r+m} - C_{2a_r+1}$ from the family $\mathcal{G}^v(a_1, a_2, \ldots, a_r; m)$. Recall that the graph belongs to $\mathcal{G}^v(a_1, a_2, \ldots, a_r; m)$. By the assumption it follows that $H$ has at least $m - a_r - 1 \geq a_r \geq 2$ vertices outside the induced cycle, i.e. adjacent to every other vertex in $H$. Let $a$ and $b$ be two such vertices and let $L$ be a set of $m$ vertices containing all vertices of the cycle and $m - 2a_r - 1$ other vertices, including $a$ but excluding $b$.

The largest clique in $H[L]$ has size $m - a_r - 1$. Let us delete the edge $\{a, b\}$ from $H$ and denote the obtained graph by $H'$. If the vertices of $H'$ are $r$-colored such that for each $i$, no copy of $K_{a_i}$ is all in color $i$, then the vertices $a$ and $b$ must have the same color. Now take $a_r$ copies of $H'$ and identify their sets $L$ leaving the remaining parts mutually disjoint. Finally, build the clique $K = K_{a_r}$ on the vertices marked by $b$. The obtained graph $G$ has $m + a_r^2$ vertices and is $K_{m-1}$-free, since its largest clique consists of the largest one in the set $L \setminus \{a\}$ enlarged by the clique $K$. Also it is easy to note that $G \to (a_1, a_2, \ldots, a_r)^v$. Indeed, consider any $r$ -coloring of the vertices of $G$ which for all $i$ does not force a clique $K_{a_i}$ in the $i$-th color class. Every copy of $H'$ has the same color on the vertices $a$ and $b$, and therefore all vertices of $K$ have the same color. $\square$

The bounds of Corollary 3 and Proposition 4 may be quite apart from each other. They get closer a little when $a_r$ is small, like in the symmetric case.

**Corollary 4.** *For every pair of integers $l \geq 2$ and $r \geq \frac{2l+1}{l-1}$*

$$(l-1)r + l + 1 \leq F^v(\underbrace{l, l, \ldots, l}_{r}; (l-1)r) \leq (l-1)r + l^2 + 1 \ . \quad \square$$

The only case where we are able to find an exact formula for $F^v(\underbrace{l, l, \ldots, l}_{r}; (l-1)r)$ is the smallest one, i.e. when $l = 2$. Note that $G \to (\underbrace{2, 2, \ldots, 2}_{r})^v$ iff $\chi(G) \geq r + 1$, where $\chi(G)$ stands for the chromatic number of a graph $G$. By Corollary 4 we have $r + 3 \leq F^v(\underbrace{2, 2, \ldots, 2}_{r \geq 5}; r) \leq r + 5$. Below we prove that the true value is $r + 5$. This seems to be a folklore result: people know it, can prove it overnight, but do not know any reference. One reason we give a complete proof of this result here is to set up such reference.

**Theorem 4.** *For every integer $r \geq 5$,*

$$F^v(\underbrace{2, 2, \ldots, 2}_{r}; r) = r + 5 \ ,$$

*i.e. a smallest $(r+1)$-chromatic, $K_r$-free graph has $r+5$ vertices.*

*Proof.*

Let $G$ be a $K_r$-free graph on $r+4$ vertices. We will show that $\chi(G) \leq r$. Let $M$ be a maximal matching in $G^c$. Since $G \not\supset K_r$, the matching $M$ consists of at least 3 edges. If $|M| \geq 4$ then it is easy to properly color $G$ with at most $r$ colors: just assign the same color to both endpoints of each edge in $M$.

This takes $|M|$ colors, and the remaining $r + 4 - 2|M|$ vertices are colored each by a diferent color. Assume thus that $M$ consists of exactly 3 edges $\{u_i, v_i\}$, $i = 1, 2, 3$. The remaining $r-2$ vertices form an independent set $I$ in $G^c$. If for some $i$, both $u_i$ and $v_i$ send an edge into $I$, then either there is another matching in $G^c$ of size 4, or there is a triangle consisting of $u_i$, $v_i$ and a vertex in $I$, and again $\chi(G) \leq r$. Let $S$ be the subset of $V(M)$ consisting of the vertices which do not send an edge into $I$. Since $G \not\supset K_r$, the induced subgraph $G^c[S]$ must be a clique., and if $|S| \geq 4$ then $\chi(G) \leq (6 - |S|) + (r-2) \leq r$. In view of the above remark, the only case left is when $S = \{v_1, v_2, v_3\}$. Then all $u_1, u_2$ and $u_3$ have the same unique neighbor $w$ in $I$. (Otherwise, again, there wou

ld be a matching of size 4.) This, in turn, implies, that there is at least one edge in the induced subgraph $G^c[u_1, u_2, u_3]$. Otherwise the set $I \setminus \{w\} \cup \{u_1, u_2, u_3\}$ would form a clique $K_r$ in $G$. Assume that $\{u_1, u_2\}$ is an edge of $G^c$. Then a proper $r$-coloring of $G$ can be obtained by coloring $v_1, v_2, v_3$ by one color, $u_1, u_2, w$ by a second color, and the remaining vertices by $r-2$ different colors. $\square$

Finally in this section, we briefly mention the classical Ramsey case, when one requires a monochromatic triangle. We are able to set bounds on the Folkman numbers $F^v(\underbrace{3, 3, \ldots, 3}_{r}; 2r)$, which for $r \geq 4$ differ only by three. We state this result without proof. The upper bound was proved in Proposition 4. The proof of the lower bound is similar but more tedious than that of Theorem 4.

**Proposition 5.** *For $r \geq 4$*

$$2r + 7 \leq F^v(\underbrace{3, 3, \ldots, 3}_{r}; 2r) \leq 2r + 10 \quad \square$$

Recently Piwakowski, Radziszowski and Urbański [9] computed that $F^v(3, 3; 4) = 14$ which is the case $r = 2$. The least known is the case $r = 3$. We only have $11 \leq F^v(3, 3, 3; 6) \leq 20$.

## 4. Smaller forbidden cliques

Note that if $m \geq 2a_r + 4$ then the graph $G$ constructed in the proof of Proposition 4 has at least two vertices adjacent to every other vertex. This allows to carry over the construction again, obtaining a graph from $\mathcal{G}^v(a_1, a_2, \dots, a_r; m-2)$. When $a_1 = \cdots = a_r = 2$, iterating this procedure $s$ times results in the following construction. Note that the graph constructed in the proof of Proposition 4 is isomorphic to $K_{r+5} - 2C_5$.

**Proposition 6.** *If $1 \leq s \leq m/3$ then $G = K_{2s+m} - sC_5$ is an $m$-chromatic graph with clique number $m - s$.*

*Proof.*

Consider the complement of $G$. It is isomorphic to the union of $s$ disjoint 5-cycles and $m - 3s$ isolated vertices. Every proper coloring of the vertices of $G$ uses $m - 3s$ colors for the isolated vertices and 3 new colors for every cycle. □

Substituting $k = s - 1$ we derive the following corollary.

**Corollary 5.** *If $0 \leq k \leq (r-2)/3$ then $F^v(\underbrace{2, 2, \dots, 2}_{r}; r + 1 - k) \leq r + 2k + 3$*

We know that for $k = 0$ and $k = 1$ this is, in fact an equality. In the case $k = 2$, we can narrow the range of posssible values to the pair $r + 6, r + 7$. One might be temped to conjecture that the Folkman numbers for $a_1 = \cdots = a_r = 2$ are always equal to $r + 2k + 3$. However, as the next result shows, this is very far from the truth, at least for larger values of $k$.

Despite the fact that the details of Theorem 5 below are very technical, the underlying idea is extremely simple. It is based on the trivial observation, that for any graph $H$ with the girth $g(H) \geq 2a_r$ and with at least $2m - 1$ vertices, where, recall, $m = \sum_{i=1}^{r}(a_i - 1) + 1$, the complement $H^c$ of $H$ satisfies the property $H^c \to (a_1, \dots, a_r)^v$. Indeed, for any $r$-coloring of $V(H)$ there is an index $i \in \{1, 2, \dots, r\}$ and a set $S$ of $2a_i - 1$ vertices of color $i$. The induced subgraph $H^c[S]$ does not contain any cycle, so it is bipartite. As such, it contains an independent set of size $\lceil |S|/2 \rceil = a_i$. If, in addition, $\alpha(H) < w$, we obtain a graph from the family $\mathcal{G}^v(a_1, a_2, \dots, a_r; w)$. For large values of $w$ the existence of such graphs can be derived via the probabilistic method (cf. [1], [2]). To get a reasonable bound on the corresponding Folkman numbers one has to estimate the order of $H$ very carefully. It turns out that a little modification of the above idea helps to push down the Folkman number by a significant amount.

**Lemma 2.** *Let $H$ be a graph with $g(H) \geq 2a_r$ and $\alpha(H) < \ell$. If $w \geq \ell$ and*

$$w - \ell + \frac{1}{2}|V(H)| \geq m ,$$

*then $G = K_{w-\ell} + H^c \in \mathcal{G}^v(a_1, a_2, \dots, a_r; w)$.*

*Proof.*

Note that the clique number of $G$ is at most $w - \ell + \alpha(H)$ which is less than $w$. Let $h : V(G) \to \{1, 2, \dots, r\}$ be an arbitrary $r$-coloring. Denote by $U$ the set of vertices of the clique $K_{w-\ell}$ and by $W$ the vertex set of $H$. By the assumption, we have

$$2|U| + |W| \geq \sum_{i=1}^{r}(2a_i - 2) + 1 .$$

Hence, there exists a color $i \in \{1, 2, \ldots, r\}$ such that

$$2|h^{-1}(i) \cap U| + |h^{-1}(i) \cap W| \geq 2a_i - 1 \ .$$

If $|h^{-1}(i) \cap U| \geq a_i$, then there is a clique $K_{a_i}$ in color $i$. Otherwise, let $T = h^{-1}(i) \cap U$, $|T| = t < a_i$, and let $S$ be a subset of $h^{-1}(i) \cap W$ such that $|S| = 2(a_i - t) - 1$. As discussed prior to Lemma 2, there is a clique of size $a_i - t$ in $H^c[S]$ which together with $T$ forms a clique $K_{a_i}$ of color $i$ in $G$. □

Armed with Lemma 2 we can now prove the following result, of an apparent asymptotic nature..

**Theorem 5.** *Let* $q = 2m - w \geq e^{e^{e^2}}$ *and* $B = 2q\frac{\log\log q}{\log q} + 2\log\log q(\log q)^{2a_r - 1}$. *If* $w \geq B$ *and* $a_r \leq \frac{\log q}{\log\log q}$, *then*

$$F^v(a_1, \ldots, a_r; w) \leq q + B \ .$$

In other words, the above result states that if we do not forbid too much (in terms of $w = m - k$) and if we do not require too much (in terms of $a_r$) then the Folkman number is bounded from above by $m + k + o(m + k)$, which is quite close to the lower bound of $m + k + 1$ determined in Proposition 1. In particular, in the symmetric case when $a_1 = \cdots = a_r = l$, $l$ fixed, the Folkman number $F^v(\underbrace{l, l, \ldots, l}_{r}; (l-1)r + 1 - k)$ is asymptotically equal to $(l-1)r + k + o(r)$. Of course, if $k = o(r)$ then one can drop $k$ from this expression. On the other hand, if, for $l = 2$, $k$ is of the order of $r$, the above bound is much better than the one established in Corollary 5.

*Proof of Theorem 5.*

We use a probabilistic argument. Let

$$n = \left\lceil \frac{q - w + (\log q)^{2a_r}}{1 - 2\frac{\log\log q}{\log q}} \right\rceil \ ,$$

$$p = 2\frac{\log q}{n} \ ,$$

and let $G(n,p)$ be a random graph on $n$ vertices, in which each edge is chosen independently with probability $p$. Let $X$ be the number of cycles of length at most $2a_r - 1$ in $G(n,p)$ and let $Y$ be the number of independent sets in $G(n,p)$ of order

$$\ell = 2\left\lfloor \frac{\log(np/2)}{p} \right\rfloor$$

Then, for the expectation of $X$, we get

$$\mathbb{E}(X) \leq \sum_{i=3}^{2a_r - 1} \binom{n}{i} \frac{(i-1)!}{2} p^i \leq \sum_{i=3}^{2a_r - 1} \frac{(np)^i}{2i} \leq \frac{1}{2}(np)^{2a_r} \ ,$$

and, from Markov's inequality, $X \leq (np)^{2a_r}$ with probability at least 1/2. On the other hand, again by Markov's inequality,

$$P(Y>0) \leq \mathbb{E}(Y) \leq \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} \leq \left[\frac{en}{\ell}\exp\left(-\frac{\ell p}{2}\right)e^{p/2}\right]^{\ell} < \frac{1}{2}$$

Combining these two facts, we infer that there exists a graph $H$ on $n$ vertices with $\alpha(H) < \ell$ and containing at most $(np)^{2a_r}$ cycles of length less than $2a_r$. By deleting one vertex from each of these cycles, we obtain a graph $H'$ on $n' = n - (np)^{2a_r}$ vertices, such that $g(H') \geq 2a_r$ and $\alpha(H) < \ell$. Note that

$$\ell \leq \frac{\log\log q}{\log q} n ,$$

$$n \leq 2(q - B + (\log q)^{2a_r})$$

and

$$n' \geq q - w + 2\frac{\log\log q}{\log q} n .$$

Thus, $w - \ell + \frac{1}{2}n' \geq m$ and $\ell \leq B \leq w$, and we can apply Lemma 2, concluding that the graph $G = K_{w-\ell} + (H')^c$ belongs to the family $\mathcal{G}^v(a_1, a_2, \ldots, a_r; w)$. It remains to observe that, since $\ell \geq \frac{\log\log q}{\log q}n - 1$ and $B\frac{\log\log q}{\log q} \geq q\left(\frac{\log\log q}{\log q}\right)^2 > 1$, we have

$$|V(G)| = n' + w - \ell \leq q - w + 2\frac{\log\log q}{\log q}n + 1 + w - \ell \leq q + \frac{\log\log q}{\log q}n + 2 \leq q + B . \quad \square$$

## 5. The most restrictive case

In this section we consider the most restrictive case of vertex-Folkman numbers, i.e. when $w = a_r + 1$. We will set upper bounds for vertex-Folkman numbers as a consequence of a recursive construction, which is a modification of the original construction of Folkman [3].

**Theorem 6.** *For all $r \geq 2$ and $2 \leq a_1 \leq \cdots \leq a_r$ the following recurrence inequality holds:*

$$F^v(a_1, a_2, \ldots, a_r; a_r+1) \leq 1 + (1 + (r-1)(F_2 - 1)) \cdot F_1 + \binom{1 + (r-1)(F_2-1)}{F_2} \cdot F_2,$$

*where $F_1 = F^v(a_1 - 1, a_2 - 1, \ldots, a_r - 1; a_r)$ and $F_2 = F^v(a_2, a_3, \ldots, a_r; a_r + 1)$.*

*Proof.*

First note that $F^v(a; a+1) = a$ and $F^v(1, \ldots, 1; 2) = 1$. Let $a_1, a_2, \ldots, a_r$ be any non-descending sequence such that $a_1 \geq 2$. We construct a graph $G \in \mathcal{G}^v(a_1, a_2, \ldots, a_r; a_r + 1)$

using the smallest graphs from the families $\mathcal{G}^v(a_1-1, a_2-1, \ldots, a_r-1; a_r)$ and $\mathcal{G}^v(a_2, a_3, \ldots, a_r; a_r+1)$.

Let $A$ be any graph of order $F_1$ belonging to $\mathcal{G}^v(a_1-1, a_2-1, \ldots, a_r-1; a_r)$ and let $B$ be any graph of order $F_2$ belonging to $\mathcal{G}^v(a_2, a_3, \ldots, a_r; a_r+1)$. By joining a vertex $x$ to a graph $H$ we mean the join operation $H+x$, defined earlier in this paper, which adds $x$ to $V(F)$ and joins $x$ to every vertex of $F$.

Let us take $1+(r-1)(F_2-1)$ disjoint copies of $A$ and join a new vertex $x$ to the union of all of them. Let $A_1, A_2, \ldots, A_{F_2}$ denote any $F_2$ of the copies of $A$. For every $j \in \{1, 2, \ldots, F_2\}$ join a new vertex $y_j$ to $A_j$, and construct a copy of the graph $B$ on the vertices $y_j$. We repeat this procedure for every $F_2$-element set of copies of $A$, each time adding a disjoint set of $F_2$ new vertices $y_j$. Let $G$ stand for the obtained graph. We claim that $G \in \mathcal{G}^v(a_1, a_2, \ldots, a_r; a_r+1)$

First we prove that $K_{a_r+1} \not\subset G$. Note that $K_{a_r} \not\subset A$ and if $u, v$ are any vertices joined to $A$ then $\{u, v\} \notin E(G)$. Thus if $K_{a_r+1} \subset G$ then $K_{a_r+1} \subset B$ but it is impossible since $B \in \mathcal{G}^v(a_2, a_3, \ldots, a_r; a_r+1)$. Thus $K_{a_r+1} \not\subset G$.

Now we prove that $G \to (a_1, a_2, \ldots, a_r)^v$. Let $h: V(G) \to \{1, 2, \ldots, r\}$ be an $r$-coloring of $G$. Assume that $h(x) = j$, $1 \le j \le r$. By definition, every copy of $A$ contains a clique $K_{a_i-1}$ of color $i$ for some $i \ne j$. There are so many copies of $A$ that at least $F_2$ of them have the monochromatic clique of the same color. More precisely, there exists $i \ne j$ such that some $F_2$ copies of $A$ contain $K_{a_i-1}$ of color $i$. Consider the vertices $y_j$ added to this set of $F_2$ copies of $A$. If one of them has color $i$, we are done. Otherwise, the corresponding graph $B$ is colored by $r-1$ colors. We know that $B \in \mathcal{G}_2 = \mathcal{G}^v(a_2, a_3, \ldots, a_r; a_r+1)$, thus for some $k \in \{1, 2, \ldots, r\}$ either $B$ contains $K_{a_k}$ of color $k$ or $B$ contains $K_{a_{k+1}}$ of color $k$. The sequence $a_1, a_2, \ldots, a_r$ is non-descending, hence $K_{a_k} \subset K_{a_{k+1}}$. Thus, one way or another, there is a copy of $K_{a_k}$ of color $k$ in $B$. □

This upper bound is quite involved, but it simplifies in the case $r=2$. Let $k, l$ be any integers such that $2 \le k \le l$.

**Corollary 6.**

$$F^v(k, l; l+1) \le 2\sum_{i=0}^{k-1} \frac{l!}{(l-i)!} - 1 \ .$$

*Proof.*

The proof is by induction on $k$ with fixed $l$. Corollary 2 implies that $F^v(2, l; l+1) \le 2l+1 = 2\sum_{i=0}^{1} \frac{l!}{(l-i)!} - 1$ for every $l \ge 2$. Assume that $k \ge 3$ and the inequality holds for

$k-1$. Using Theorem 6 we obtain

$$F^v(k,l;l+1) \leq 1 + lF^v(k-1,l-1;l) + l \leq 1 + l\left(2\sum_{i=0}^{k-2}\frac{(l-1)!}{(l-1-i)!}\right) =$$
$$= -1 + 2 + 2l!\sum_{i=0}^{k-2}\frac{1}{(l-i-1)!} = -1 + 2l!\left(\frac{1}{l!} + \sum_{i=0}^{k-2}\frac{1}{(l-i-1)!}\right) =$$
$$= -1 + 2l!\sum_{i=-1}^{k-2}\frac{1}{(l-i-1)!} = -1 + 2\sum_{i=0}^{k-1}\frac{l!}{(l-i)!} \quad \square$$

When $k = l$, the upper bound of Corollary 6 assumes the following simple form:

**Corollary 7.**

$$F^v(k,k;k+1) \leq \lfloor 2k!(e-1)\rfloor - 1.$$

*Proof.*

Note that $\sum_{i=0}^{k-2}\frac{k!}{(k-i)!} = k!\sum_{i=2}^{k}\frac{1}{i!}$ $\square$

In particular, Corollary 7 yields that $F^v(3,3;4) \leq 19$. As mentioned before, it was proved in [9] that this Folkman number is equal to 14 (see also [11]).

## 6. Edge-Folkman numbers

In this section we show a connection between vertex- and edge-Folkman numbers. We begin with some definitions. Assume that $a_1, a_2, \ldots, a_r$ is a non-descending sequence of integers greater than 1. We write $G \to (a_1, a_2, \ldots, a_r)^e$ if every $r$-coloring of the edges of a graph $G$ forces a complete subgraph $K_{a_i}$ of color $i$ for some $i \in \{1, 2, \ldots, r\}$. For $w > a_r$ let

$$\mathcal{G}^e(a_1, a_2, \ldots, a_r; w) = \{G : G \to (a_1, a_2, \ldots, a_r)^e \text{ and } K_w \not\subseteq G\}.$$

We define the *edge-Folkman numbers* as follows:

$$F^e(a_1, a_2, \ldots, a_r; w) = \min\{|V(G)| : G \in \mathcal{G}^e(a_1, a_2, \ldots, a_r; w)\}.$$

The existence of the edge-Folkman numbers $F^e(a_1, a_2; w)$ was proved by Folkman [3] and his result was generalized to arbitrary $r$ by Nešetřil and Rödl [8]. Of course

$$F^e(a_1, a_2, \ldots, a_r; w) = R(a_1, a_2, \ldots, a_r)$$

for every $w > R(a_1, a_2, \ldots, a_r)$, where $R(a_1, a_2, \ldots, a_r)$ is the Ramsey number. Now we present a simple observation, which will be useful in linking edge- and vertex-Folkman numbers. The symmetric case of the lemma below was proved in [5] for two colors and in [11] for arbitrarily many colors. The idea of the proof is basically taken from [5].

**Lemma 3.** *Let* $R_i = R(a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_r)$ *for every* $i \in \{1, 2, \ldots, r\}$. *If* $H \in \mathcal{G}^v(R_1, R_2, \ldots, R_r; w)$, *then* $H + v \in \mathcal{G}^e(a_1, a_2, \ldots, a_r; w + 1)$.

*Proof.*

Suppose that the edges of $H + v$ are $r$-colored arbitrarily. We color every vertex $x \in H$ with the color of the edge $\{v, x\}$. There exists $i$ $(1 \le i \le r)$ such that $H$ contains a complete subgraph $K_{R_i}$ with all vertices of color $i$. If $K_{R_i}$ contains $K_{a_i - 1}$ with all edges of color $i$, then this clique $K_{a_i-1} + v$ creates a complete graph $K_{a_i}$ with all edges of color $i$. If it does not, then there exists $j \neq i$ $(1 \le j \le r)$ such, that $K_{R_i}$ contains $K_{a_j}$ with all edges of color $j$, since $K_{R_i} \to (a_1, \ldots, a_i - 1, \ldots, a_r)^e$. Thus every $r$-coloring of the edges of $H + v$ forces a complete subgraph $K_{a_l}$ of color $l$ for some $l \in \{1, 2, \ldots, r\}$. □

The above lemma allows to bound edge-Folkman numbers by existing bounds on the respective vertex-Folkman numbers increased by one. The upper bounds presented below follow immediately from Corollaries 6 and 7 and from Lemma 3.

**Corollary 8.** *For* $k, l \ge 3$, *let* $M = \max\{R(k-1, l), R(k, l-1)\}$ *and* $m = \min\{R(k-1, l), R(k, l-1)\}$. *Then*

$$F^e(k, l; M + 2) \le 2 \sum_{i=0}^{m-1} \frac{M!}{(M-i)!} \ .$$

*In particular,*

$$F^e(k, k; M + 2) \le \lfloor 2M!(e - 1) \rfloor \ ,$$

*where* $M = R(k, k-1)$. □

Unfortunately, the bounds obtained that way are not very good. For example we get $F^e(3, 3; 5) \le 20$, $F^e(3, 4; 8) \le 314$, $F^e(4, 4; 11) \le \lfloor 2 \cdot 9!(e - 1) \rfloor = 1\ 247\ 060$, and $F^e(5, 5; 27) \le \lfloor 2 \cdot 25!(e - 1) \rfloor = 3.33 \cdot 10^{25}$. It turns out that only the first bound is reasonably close to the truth. It has been proved recently in [9] that $F^e(3, 3; 5) = 15$. (For the peculiar history of the struggle to determine this Folkman number see [11].) The number $F^e(3, 3; 6) = 8$ is one of very few Folkman numbers known besides $F^e(3, 3; 5)$ (cf. Graham [4]). Spencer [10] proved that $F^e(3, 3; 4) < 10^{10}$. Although at the very first glance this bound is not very impressive, yet this achievement was awarded by one of Erdős' prizes.

## Appendix

**Proposition 7.** *Let* $G$ *be a stable graph and let* $S$ *be a maximal independent set in* $G$. *Then there exists a matching saturating all vertices in* $S$.

*Proof.*

Let $M$ be a matching saturating as many vertices in $S$ as it is possible and let $v$ be a vertex in $S$ not saturated by $M$. Assume that $S'$ is a maximal independent set in $G - v$ (the graph obtaining by deleting $v$ from $G$) and put $U = S' \cap S$ and $T = S' \cap (V \setminus S)$. Of course $T$ is an independent set in $G$. Note that $|N(T) \cap S| = |T|$. Indeed, if $|N(T) \cap S| < |T|$ then $[(S \setminus N(T)) \cap S] \cup T$ is the independent set greater than $S$. On the other hand, if

$|N(T) \cap S| > |T|$ then $|U \cup (N(T) \cap S)| > |U \cup T|$ and then $|S| > |S'|$ - a contradiction, since $G$ is stable. Note also that $v \in N(T) \cap S$, otherwise $S' \cup \{v\}$ is an independent set greater than $S$. Since $S$ is maximal, we have also $|N(T') \cap S| \geq |T'|$ for every $T' \subset T$, because $|U \cup S \setminus (N(T') \cap S)|$ is not greater than $|S|$. Thus, from Hall's theorem, there exists a perfect matching $M'$ in $G[(N(T) \cap S) \cup T]$. Let $L$ be the set of the edges from $M$ which saturate $S \cap N(T)$. Note that $(M \setminus L) \cup M'$ is a matching which saturates the vertex $v$ as well as all vertices in $S$ saturated by $M$. This is a contradiction with the choice of $M$. □

**Lemma 1.** *Let $G$ be a stable graph of order $n$. Then*

(i) $\alpha(G) < n/2$

(ii) *if $n = 2k+1$ and $\alpha(G) = k$ then $G$ is isomorphic to the cycle $C_n$*

*Proof.*

(i) from Proposition 7 we have $\alpha(G) \leq n/2$. Suppose that $\alpha(G) = n/2$. Let $M$ be a matching which saturates a maximal independent set. Note that removing both endpoints of any edge from $M$ decreases the independence number $\alpha(G)$, but this is a contradiction, since $G$ is stable.

(ii) Assume that $n = 2k+1$. Let $S$ be a maximal independent set in $G$ and $M$ be a matching saturating $S$. Denote by $w$ the vertex not saturated by $M$. We can assume that $w$ has a neighbor in $V \setminus S$. Indeed, supose that $N(w) \cap (V \setminus S) = \emptyset$. Let $v$ be a neighbor of $w$ in $S$. The vertex $v$ exists since $S$ is maximal. Let $e = \{u, v\}$ be the edge in $M$ incident to $v$ and let $S'$ be a maximal independent set in the graph $G'$ obtained by deleting the vertices $u$ and $v$ from $G$. Assume also $T = S' \cap (V \setminus S)$. The matching $M$ saturates $S'$ since $w \notin N(T)$, hence we have the following situation: $S'$ is a maximal independent set in $G'$, $M$ saturates $S'$, $w$ is the vertex not saturated by $M$ and it has the neighbor $v$ in $V \setminus S'$. Thus without loss of generality we can assume that $w$ has a neighbor $v$ in $V \setminus S$. Let $v'$ be the vertex joined with $v$ by the edge $e \in M$ and let $S'$ be a maximal independent set in the graph $G'$ obtained by deleting the vertices $v$ and $v'$ from $G$. Put $T = S' \cap (V \setminus S)$ and note that $|N(T) \cap S| = |T|$ and $v' \in N(T) \cap S$, similarly as in proof of Proposition 2. We know that there are at most $k-1$ vertices in $S'$ saturated by $M$, thus $w \in T$ since $w$ is the only vertex not saturated by $M$. Moreover, since $S$ is maximal, for every subset $T'$ of $T$ we have $|N(T') \cap S| \geq |T'|$. Thus from Hall's theorem there exists a perfect matching in the subgraph of $G$ induced by the set $T \cup (N(T) \cap S)$. But this means that $M$ is not maximal since it does not saturate the vertex $w$. Thus from Berge's theorem, the vertices $w, v'$ are connected by an $M$-augmenting path. This path together with the edges $\{v, v'\}$ and $\{w, v\}$ creates an odd cycle $C$. It is easy to check that $C$ must be hamiltonian. Indeed, suppose to the contrary that $C$ is not hamiltonian. Then since $C$ is odd, there are at least two vertices outside the cycle. We know that all such vertices are saturated by $M$, hence there exists at least one edge $e' \in M$ with both endpoints outside the cycle $C$. Note that deleting both endpoints of $e'$ decreases the number $\alpha(G)$ – a contradiction. Finally suppose that $C$ has a chord $\{x, y\}$. Then $C$ consists of two paths connecting $x$ and $y$. Consider the odd one. If we delete both endpoints of any of its edges not adjacent to $\{x, y\}$ then $\alpha(G)$ decreases again. Thus $G \cong C_n$. □

## References


1. N.Alon, J.H.Spencer, *The probabilistic method*, 1992.
2. P.Erdős, *Graph coloring and probability*, Canadian Journal of Mathematics **11** (1959), 24–38.
3. J.Folkman, *Graphs with monochromatic complete subgraphs in every edge coloring*, SIAM J. Appl. Math. **18** (1970), 19–24.
4. R.L.Graham, *On edgewise 2-colored graphs with monochromatic triangles and containing no complete hexagon*, J. Combin. Theory **4** (1968), 300.
5. R.W.Irving, *On a bound of Graham and Spencer for graph-coloring constant*, J. Combin. Theory **15** (1973), 200–203.
6. S.Lin, *On Ramsey numbers and $K_r$-coloring of graphs*, J. Combin. Theory **B 12** (1972), 82–92.
7. T.Łuczak, S.Urbański, *A note on restricted vertex Ramsey numbers*, Periodica Mathematica Hungarica **33** (1996), 101–103.
8. J.Nešetřil, V.Rödl, *The Ramsey property for graphs with forbidden complete subgraphs*, J. Combin. Theory **B 20** (1976), 243–249.
9. K.Piwakowski, S.P.Radziszowski, S.Urbański, *Computation of the Folkman number* $F(3,3;5)$, in preparation.
10. J.H.Spencer, *Three hundred million points suffice*, J. Combin. Theory (a) **49** (1988), 210–217. See erratum in vol. **50** p.323.
11. S.Urbański, *Remarks on 15-vertex (3,3)-Ramsey graphs not containing* $K_5$, Disc. Math. Graph Theory **33** (1996), 173–179.