# The Generalization of Dirac's Theorem for Hypergraphs

Endre Szemerédi[1], Andrzej Ruciński[2,⋆], and Vojtěch Rödl[3,⋆⋆]

[1] Rutgers University, New Brunswick
szemered@cs.rutgers.edu
[2] A. Mickiewicz University, Poznań, Poland
rucinski@amu.edu.pl
[3] Emory University, Atlanta, GA
rodl@mathcs.emory.edu

## 1 Introduction and Main Result

A substantial amount of research in graph theory continues to concentrate on the existence of hamiltonian cycles and perfect matchings. A classic theorem of Dirac states that a sufficient condition for an $n$-vertex graph to be hamiltonian, and thus, for $n$ even, to have a perfect matching, is that the minimum degree is at least $n/2$. Moreover, there are obvious counterexamples showing that this is best possible.

The study of hamiltonian cycles in hypergraphs was initiated in [1] where, however, a different definition than the one considered here was introduced. Given an integer $k \geq 2$, a $k$-uniform hypergraph is a hypergraph (a set system) where every edge (set) is of size $k$.

By *a cycle* we mean a $k$-uniform hypergraph whose vertices can be ordered cyclically $v_1, \ldots, v_l$ in such a way that for each $i = 1, \ldots, l$, the set $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\}$ is an edge, where for $h > l$ we set $v_h = v_{h-l}$ . *A hamiltonian cycle* in a $k$-uniform hypergraph $H$ is a spanning cycle in $H$, that is, a sub-hypergraph of $H$ which is a cycle and contains all vertices of $H$. A $k$-uniform hypergraph containing a hamiltonian cycle is called *hamiltonian*.

This notion and its generalizations have a potential to be applicable in many contexts which still need to be explored. An application in the relational database theory can be found in [2]. As observed in [5], the square of a (graph) hamiltonian cycle naturally coincides with a hamiltonian cycle in a hypergraph built on top of the triangles of the graph. More precisely, given a graph $G$, let $Tr(G)$ be the set of triangles in $G$. Define a hypergraph $H^{Tr}(G) = (V(G), Tr(G))$. Then there is a one-to-one correspondence between hamiltonian cycles in $H^{Tr}(G)$ and the squares of hamiltonian cycles in $G$. For results about the existence of squares of hamiltonian cycles see, e.g., [6].

As another potential application consider a seriously ill patient taking 24 different pills on a daily basis, one at a time every hour. Certain combinations

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of three pills can be deadly if taken within 2.5 hour. Let $D$ be the set of deadly triplets of pills. Then any safe schedule corresponds to a hamiltonian cycle in the hypergraph which is precisely the complement of $D$.

A natural extension of Dirac's theorem to $k$-graphs, $k \geq 2$, has been conjectured in [5], where as a sufficient condition one demands that every $(k-1)$-element set of vertices is contained in at least $\lfloor n/2 \rfloor$ edges. The following construction of a $k$-uniform hypergraph $H_0$, also from [5], shows that the above conjecture, if true, is nearly best possible (best possible for $k = 3$).

Let $V = V' \cup \{v\}$, $|V| = n$. Split $V' = X \cup Y$, where, $|X| = \lfloor \frac{n-1}{2} \rfloor$ and $|Y| = \lceil \frac{n-1}{2} \rceil$. The edges of $H_0$ are all $k$-element subsets $S$ of $V$ such that $|X \cap S| \neq \lfloor \frac{k}{2} \rfloor$ or $v \in S$. It is shown in [5] that $H_0$ is not hamiltonian, while every $(k-1)$-element set of vertices belongs to at least $\lfloor \frac{n-k+1}{2} \rfloor$ edges.

In [9] we proved an approximate version of the conjecture from [5] for $k = 3$, and in [11] we give a generalization of that result to $k$-uniform hypergraphs for arbitrary $k$.

**Theorem 1 ([11]).** *Let $k \geq 3$ and $\gamma > 0$. Then, for sufficiently large $n$, every $k$-uniform hypergraph on $n$-vertices such that each $(k-1)$-element set of vertices is contained in at least $(1/2 + \gamma)n$ edges is hamiltonian.*

## 2 The Idea of Proof

The idea of the proof is as follows. As a preliminary step, we find in $H$ a powerful path $A$, called *absorbing* which has the property that *every* not too large subset of vertices can be "absorbed" by that path. We also put aside a small subset of vertices $R$ which preserves the degree properties of the entire hypergraph.

On the sub-hypergraph $H' = H - (A \cup R)$ we find a collection of long, disjoint paths which cover almost all vertices of $H'$. Then, using $R$ we "glue" them and the absorbing path $A$ together to form a long cycle in $H$. In the final step, the vertices which are not yet on the cycle are absorbed by $A$ to form a hamiltonian cycle in $H$.

The main tool allowing to cover almost all vertices by disjoint paths is a generalization of the regularity lemma from [12].

Given a $k$-uniform hypergraph $H$ and $k$ non-empty, disjoint subsets $A_i \subset V(H)$, $i = 1, \ldots, k$, we define $e_H(A_1, \ldots, A_k)$ to be the number of edges in $H$ with one vertex in each $A_i$, and the *density* of $H$ with respect to $(A_1, \ldots, A_k)$ as

$$d_H(A_1, \ldots, A_k) = \frac{e_H(A_1, \ldots, A_k)}{|A_1| \cdots |A_k|}.$$

A $k$-uniform hypergraph $H$ is *$k$-partite* if there is a partition $V(H) = V_1 \cup \cdots \cup V_k$ such that every edge of $H$ intersects each set $V_i$ in precisely one vertex. For a $k$-uniform, $k$-partite hypergraph $H$, we will write $d_H$ for $d_H(V_1, \ldots, V_k)$ and call it the *density of $H$*.

We say that a $k$-uniform, $k$-partite hypergraph $H$ is *$\epsilon$-regular* if for all $A_i \subseteq V_i$ with $|A_i| \geq \epsilon |V_i|$, $i = 1, \ldots, k$, we have

$$|d_H(A_1, \ldots, A_k) - d_H| \leq \epsilon.$$

The following result, called *weak regularity lemma* as opposed to the stronger result in [4], is a straightforward generalization of the graph regularity lemma from [12].

**Lemma 1 (Weak regularity lemma for hypergraphs).** *For all $k \geq 2$, every $\epsilon > 0$ and every integer $t_0$ there exist $T_0$ and $n_0$ such that the following holds. For every $k$-uniform hypergraph $H$ on $n > n_0$ vertices there is, for some $t_0 \leq t \leq T_0$, a partition $V(H) = V_1 \cup \cdots \cup V_t$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_t| \leq |V_1| + 1$ and for all but at most $\epsilon t^k$ sets of partition classes $\{V_{i_1}, \ldots, V_{i_k}\}$, the induced $k$-uniform, $k$-partite sub-hypergraph $H[V_{i_1}, \ldots, V_{i_k}]$ of $H$ is $\epsilon$-regular.*

The above regularity lemma, combined with the fact that every dense $\epsilon$-regular hypergraph contains an almost perfect path-cover, yields an almost perfect path-cover of the entire hypergraph $H$.

## 3 Results for Matchings

A perfect matching in a $k$-uniform hypergraph on $n$ vertices, $n$ divisible by $k$, is a set of $n/k$ disjoint edges. Clearly, every hamiltonian, $k$-uniform hypergraph with the number of vertices $n$ divisible by $k$ contains a perfect matching.

Given a $k$-uniform hypergraph $H$ and a $(k-1)$-tuple of vertices $v_1, \ldots, v_{k-1}$, we denote by $N_H(v_1, \ldots, v_{k-1})$ the set of vertices $v \in V(H)$ such that $\{v_1, \ldots, v_{k-1}, v\} \in H$. Let $\delta_{k-1}(H) = \delta_{k-1}$ be the minimum of $|N_H(v_1, \ldots, v_{k-1})|$ over all $(k-1)$-tuples of vertices in $H$.

For all integer $k \geq 2$ and $n$ divisible by $k$, denote by $t_k(n)$ the smallest integer $t$ such that every $k$-uniform hypergraph on $n$ vertices and with $\delta_{k-1} \geq t$ contains a perfect matching.

For $k = 2$, that is, in the case of graphs, we have $t_2(n) = n/2$. Indeed, the lower bound is delivered by the complete bipartite graph $K_{n/2-1, n/2+1}$, while the upper bound is a trivial corollary of Dirac's condition [3] for the existence of Hamilton cycles.

In [10] we study $t_k$ for $k \geq 3$. As a by-product of our result about hamiltonian cycles in [11] (see Theorem 2 above), it follows that $t_k(n) = n/2 + o(n)$. Kühn and Osthus proved in [7] that

$$\frac{n}{2} - k + 1 \leq t_k(n) \leq \frac{n}{2} + 3k^2 \sqrt{n \log n}.$$

The lower bound follows by a simple construction, which, in fact, for $k$ odd yields $t_k(n) \geq n/2 - k + 2$. For instance, when $k = 3$ and $n/2$ is an odd integer, split the vertex set into sets $A$ and $B$ of size $n/2$ each, and take as edges all triples of vertices which are either disjoint from $A$ or intersect $A$ in precisely two elements.

In [10] we improve the upper bound from [7].

**Theorem 2.** *For every integer $k \geq 3$ there exists a constant $C > 0$ such that for sufficiently large $n$,*

$$t_k(n) \leq \frac{n}{2} + C \log n.$$

It is very likely that the true value of $t_k(n)$ is yet closer to $n/2$. Indeed, in [5] it is conjectured that $\delta_{k-1} \geq n/2$ is sufficient for the existence of a Hamilton cycle, and thus, when $n$ is divisible by $k$, the existence of a perfect matching. Based on this conjecture and on the above mentioned construction from [7], we believe that $t_k(n) = n/2 - O(1)$. In fact, for $k = 3$, we conjecture that $t_3(n) = \lceil n/2 \rceil - 1$.

Our belief that $t_k(n) = n/2 - O(1)$ is supported by some partial results. For example, we are able to show that the threshold function $t_k(n)$ has a stability property, in the sense that hypergraphs that are "away" from the "extreme case" $H_0$, described in Section 1, contain a perfect matching even when $\delta_{k-1}$ is smaller than but not too far from $n/2$.

Interestingly, if we were satisfied with only a partial matching, covering all but a constant number of vertices, then this is guaranteed already with $n/2+o(n)$ replaced by $n/k$, that is, when $\delta_{k-1} \geq n/k$.

We have also another related result, about the existence of a fractional perfect matching, which is a simple consequence of Farkas' Lemma (see, e.g.,[8]). *A fractional perfect matching* in a $k$-uniform hypergraph $H = (V, E)$ is a function $w : E \to [0, 1]$ such that for each $v \in V$ we have

$$\sum_{e \ni v} w(e) = 1.$$

In particular, it follows from our result that if $\delta_{k-1}(H) \geq n/k$ then $H$ has a fractional perfect matching, so, again, the threshold is much lower than that for perfect matchings.

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