Perfect matchings in uniform hypergraphs with large minimum degree.

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Abstract

A perfect matching in a k-uniform hypergraph on n vertices, n divisible by k, is a set of n/k disjoint edges. In this paper we give a sufficient condition for the existence of a perfect matching in terms of a variant of the minimum degree. We prove that for every $k \geq 3$ and sufficiently large n, a perfect matching exists in every n-vertex k-uniform hypergraph in which each set of $k-1$ vertices is contained in $n/2 + \Omega(\log n)$ edges. Owing to a construction in [7], this is nearly optimal. For almost perfect and fractional perfect matchings we show that analogous thresholds are close to n/k rather than $n/2$.

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1 Introduction

Given a k-uniform hypergraph H and a $(k-1)$-tuple of vertices $v_1, \ldots, v_{k-1}$, we denote by $N_H(v_1, \ldots, v_{k-1})$ the set of vertices $v \in V(H)$ such that $\{v_1, \ldots, v_{k-1}, v\} \in H$. Let $\delta_{k-1}(H) = \delta_{k-1}$ be the minimum of $|N_H(v_1, \ldots, v_{k-1})|$ over all $(k-1)$-tuples of vertices in H.

For all integers $k \geq 2$ and n divisible by k, denote by $t_k(n)$ the smallest integer t such that every k-uniform hypergraph on n vertices and with $\delta_{k-1} \geq t$ contains a *perfect matching*, that is a set of n/k disjoint edges.

For $k = 2$, that is, in the case of graphs, we have $t_2(n) = n/2$. Indeed, the lower bound is delivered by the complete bipartite graph $K_{n/2-1,n/2+1}$, while the upper bound is a trivial corollary of Dirac's condition for the existence of Hamilton cycles (there is also an easy direct argument – see Proposition 2.1 below).

The main goal of this paper is to study $t_k(n)$ for $k \geq 3$. As a by-product of a result about Hamilton cycles in [13], it follows that $t_k(n) = n/2 + o(n)$. Kühn and Osthus proved in [7] that

$$\frac{n}{2} - 2\left\lfloor \frac{k}{2} \right\rfloor + 1 \leq t_k(n) \leq \frac{n}{2} + 3k^2\sqrt{n \log n}. \qquad (1)$$

The lower bound follows by a simple construction. For instance, when $k = 3$ and $n/2$ is an odd integer, split the vertex set into sets A and B of size $n/2$ each and take as edges all triples of vertices which are either disjoint from A or intersect A in precisely two elements (see Figure 1).

For the upper bound, Kühn and Osthus used the probabilistic method and a reduction to the k-partite case. By employing 'the method of absorption', first used in [10] in the context of Hamilton (hyper)cycles, we improve the upper bound, replacing the term $O(\sqrt{n \log n})$ by $O(\log n)$.

Theorem 1.1 *For every integer $k \geq 3$ there exists a constant $C > 0$ such that for sufficiently large n,*

$$t_k(n) \leq \frac{n}{2} + C \log n.$$

Remark 1.1 It is very likely that the true value of $t_k(n)$ is yet closer to $n/2$. Indeed, in [13] it is conjectured that $\delta_{k-1} \geq n/2$ is sufficient for the existence of a tight Hamilton cycle ('tight' means here that every k consecutive vertices form an edge). When n is divisible by k, such a cycle, clearly, contains a

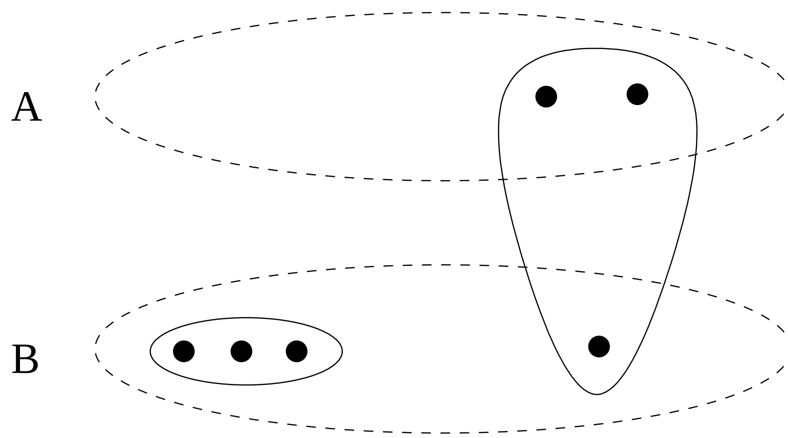

Figure 1: A 3-uniform hypergraph $H_3(n)$ with $\delta_2 = n/2 - 2$ and no perfect matching ($|A| = |B| = n/2$ is an odd integer).

perfect matching. Based on this conjecture and on the above mentioned construction from [7], we believe that $t_k(n) = n/2 - O(1)$. In fact, for $k = 3$, a proof in [11] (which is still work in progress) suggests that already $\delta_2 \geq n/2 - 1$ guarantees a tight Hamilton path, which, again, for n divisible by k, yields a perfect matching. Hence, in view of (1) it is reasonable to conjecture that $t_3(n) = \lceil n/2 \rceil - 1$.

Remark 1.2 Our belief that $t_k(n) = n/2 - O(1)$ is supported by some partial results. For example, we can show that the threshold function $t_k(n)$ has a stability property, in the sense that hypergraphs that are "away" from an "extreme case" contain a perfect matching even when δ_{k-1} is smaller than but not too far from $n/2$.

More precisely, let $H_k = H_k(n)$ be the k-uniform hypergraph on n vertices, described in [7], which yields the lower bound on $t_k(n)$. Then *for every* $\varepsilon > 0$ *there exists* $\gamma > 0$ *such that whenever* $\delta_{k-1}(H) > (1/2 - \gamma)n$ *and for every copy* H' *of* H, *with* $V(H') = V(H_k)$, *we have* $|E(H') \setminus E(H_k)| > \varepsilon n^k$, *then* H *contains a perfect matching.* This and other related results will appear in [12].

Interestingly, if we were satisfied with an 'almost perfect matching', which covers all but rk vertices, where $r \geq 1$ is fixed, then this is guaranteed already by the condition $\delta_{k-1} \geq c(r,k)n$, where $c(r,k) = 1/k$ for $r \geq k - 2$

and $c(r,k) < 1/2$ for all $r \geq 1$ (see Propositions 2.1 and 2.2 in Section 2.1). The fact that an almost perfect matching appears already when $\delta_{k-1}(H)$ is significantly smaller than $n/2$, plays a crucial role in our proof of Theorem 1.1.

In the case when $r \geq k-2$, Kühn and Osthus in [7] obtained an analogous result about almost perfect matchings in k-partite k-uniform hypergraphs. However, for general k-uniform hypergraphs, they have, similarly as in (1), an additive $O(\sqrt{n \log n})$ term. Moreover, Kühn and Osthus [7] gave examples showing that n/k is essentially best possible.

In the last section we present some results about the existence of fractional perfect matchings in k-uniform hypergraphs, which are a simple consequence of Farkas' Lemma (see, e.g., [3] or [8]). *A fractional perfect matching* in a k-uniform hypergraph $H = (V, E)$ is a function $w : E \to [0, 1]$ such that for each $v \in V$ we have

$$\sum_{e \ni v} w(e) = 1.$$

It follows that if an n-vertex k-uniform hypergraph has a fractional perfect matching then

$$\sum_{e \in H} w(e) = \frac{n}{k}. \tag{2}$$

In particular, we prove that if $\delta_{k-1}(H) \geq n/k$ then H has a fractional perfect matching, so, again, the threshold is much lower than that for perfect matchings. Moreover, this is optimal in the sense that there exists an n-vertex k-uniform hypergraph with $\delta_{k-1}(H) = \lceil n/k \rceil - 1$ which has no fractional perfect matching.

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2 Proof of Theorem 1.1

2.1 Almost perfect matchings

We first prove a simple result guaranteeing an 'almost perfect matching' already when δ_{k-1} is close to n/k. Let $\beta(H)$ denote the size of a largest matching in H and, for $r = 1, 2, \ldots$, let $t_k^{(r)}(n)$ be the smallest integer t such that for every k-uniform hypergraph H on n vertices and with $\delta_{k-1}(H) \geq t$ we have $\beta(H) \geq n/k - r$.

Proposition 2.1 *For all integers $k \geq 2$, $r \geq k-2$ and n divisible by k,*

$$t_k^{(r)}(n) = \frac{n}{k} - r.$$

Proof: The lower bound $t_k^{(r)}(n) \geq n/k - r$, true in fact for all $r \geq 1$, is a consequence of the following construction provided by Kühn and Osthus in [7] (Lemma 17 with $q = r+1$). Let us split the vertex set into an $(n/k - r - 1)$-element set A and an $(n - |A|)$-element set B, and take as edges all k-element sets of vertices which intersect A. We have $\delta_{k-1} = |A|$, but, on the other hand, the size of any matching is at most $|A|$.

For the upper bound, we only give the proof in the case $r = k-2$. For $r \geq k-2$ the proof is practically the same.

Let M be a largest matching in H and suppose that M misses at least $k(k-1)$ vertices, that is, $|M| \leq n/k - k + 1$. Let us arbitrarily group these vertices into k disjoint sets $f_1, \ldots, f_k$ of size $k-1$. Each set f_i is contained in at least $n/k - k + 2$ edges of H whose k-th vertices are all in $V(M)$. Altogether, the sets $f_1, \ldots, f_k$ send at least $k(n/k - k + 2)$ edges into M, and thus, by averaging, there is an edge e in M which receives at least

$$\left\lceil \frac{k(n/k - k + 2)}{|M|} \right\rceil \geq \left\lceil \frac{k(n/k - k + 2)}{n/k - k + 1} \right\rceil \geq k + 1$$

of these edges. But this means that there are two distinct vertices $u_1, u_2 \in e$ and two (disjoint) sets f_{i_1} and f_{i_2} such that $e_j = f_{i_j} \cup \{u_j\} \in H$, $j = 1, 2$. Replacing e by e_1 and e_2 yields a larger matching than M – a contradiction (see Figure 2). ∎

An open problem that remains is to determine $t_k^{(r)}(n)$ for $1 \leq r \leq k-3$. Observe that, in view of Proposition 2.1, the smallest unknown instance is

$t_4^{(1)}(n)$. We have only a partial result in this direction, which shows, nevertheless, that already for $r = 1$ the parameter $t_k^{(r)}(n)$ is substantially smaller than $n/2$. For $k \geq 4$, let

$$c(r,k) = \begin{cases} \frac{k^2-k+2}{2k^2} & r = 1, \\ \frac{k-1}{(r+1)k} & 2 \leq r \leq k-3. \end{cases} \tag{3}$$

Note that we have $1/k < c(r,k) < 1/2$.

Proposition 2.2 *For all $k \geq 4$ and $1 \leq r \leq k-3$, we have*

$$t_k^{(r)}(n) \leq c(r,k)n.$$

Proof: We only give the proof in the most interesting case $r = 1$, leaving the similar proof in the general case for the reader. Set $c = c(1,k)$, and assume that $\delta_{k-1}(H) \geq cn$ but $\beta(H) \leq n/k - 2$. Let M be a largest matching in H, and let S be the set of vertices not covered by M. Then, $s = |S| \geq 2k$, and for every $(k-1)$-tuple of vertices $f \in \binom{S}{k-1}$, we have $|N_H(f)| \geq cn$, and, due to the maximality of M, $N_H(f) \subseteq V(M)$. Thus, by averaging, there is an edge $e_0 \in M$ such that the set of edges

$$E_0 = \{e \in H : |e \cap e_0| = 1, |e \cap S| = k-1\}$$

has size

$$|E_0| \geq \frac{\binom{s}{k-1}cn}{|M|} > \binom{s}{k-1}ck. \tag{4}$$

Let us partition $\binom{S}{k-1} = \bigcup_{i=0}^{k} A_i$, where

$$A_i = \left\{ f \in \binom{S}{k-1} : |\{e \in E_0 : e \supset f\}| = i \right\}.$$

Further, let $B = \bigcup_{i=2}^{k} A_i$. Then

$$|A_0| + |A_1| + |B| = \binom{s}{k-1} \tag{5}$$

and $\sum_{i=1}^{k} i|A_i| = |E_0|$, yielding, by (4),

$$|A_1| + k|B| > \binom{s}{k-1}ck. \tag{6}$$

Note that for all $f \in B$ and $g \in A_1 \cup B$, we have $f \cap g \neq \emptyset$, since otherwise M could be enlarged. Thus, for every $f \in B$, all $\binom{s-k+1}{k-1}$ $(k-1)$-tuples $g \subset S \setminus f$ belong to A_0. Since every $g \in A_0$ is counted here at most $\binom{s-k+1}{k-1}$ times, we conclude that $|A_0| \geq |B|$.

Using this fact, recalling that $s \geq 2k$, and subtracting (5) from (6), we infer that

$$|B| > \frac{1}{k-2}\binom{s}{k-1}(ck-1) \geq \binom{s}{k-1}\frac{k-1}{2k} \geq \binom{s-1}{k-2}.$$

However, by the Erdős-Ko-Rado theorem (see [4]), this means that there are two disjoint $(k-1)$-tuples in B, and M can be enlarged – a contradiction. ■

It is interesting to note that the same proof yields the following result. Let $k \geq 3$ and $n = k-1 \pmod{k}$. If

$$\delta_{k-1}(H) \geq \left(\frac{1}{2} - \frac{k-2}{2k(2k-1)}\right) n,$$

then $\beta(H) = \lfloor n/k \rfloor$. Hence, there is in H a matching as perfect as it gets, already when $\delta_{k-1}(H)$ is well below $n/2$. For instance, when $n = 3m+2$, then a matching of size m is guaranteed already by $\delta_2 \geq 7n/15$. On the other hand, by (1), we know that for $n = 3m$, $\delta_2 \sim n/2$ is the threshold for the presence of a perfect matching. In the case $n = 3m+1$ it remains open whether a matching of size m is guaranteed by $\delta_2 \geq cn$ for some $c < 1/2$.

2.2 The idea of the proof of Theorem 1.1

We first come up with an absorption device allowing to include outstanding vertices into an existing matching.

Definition 2.1 For a k-tuple of vertices W, we call an edge $e \in H$ *friendly (with respect to W)* if $e \cap W = \emptyset$ and there are vertices $u_0 \in e$ and $w_0 \in W$ such that $e_1 = e \setminus \{u_0\} \cup \{w_0\} \in H$ and $e_2 = W \setminus \{w_0\} \cup \{u_0\} \in H$ (see Figure 3).

The concept of a friendly edge will be used in the following context. For a given matching M, if the vertices of W are outside M, while e is an edge of M which is friendly with respect to W, then M can be enlarged by replacing e with the edges e_1 and e_2.

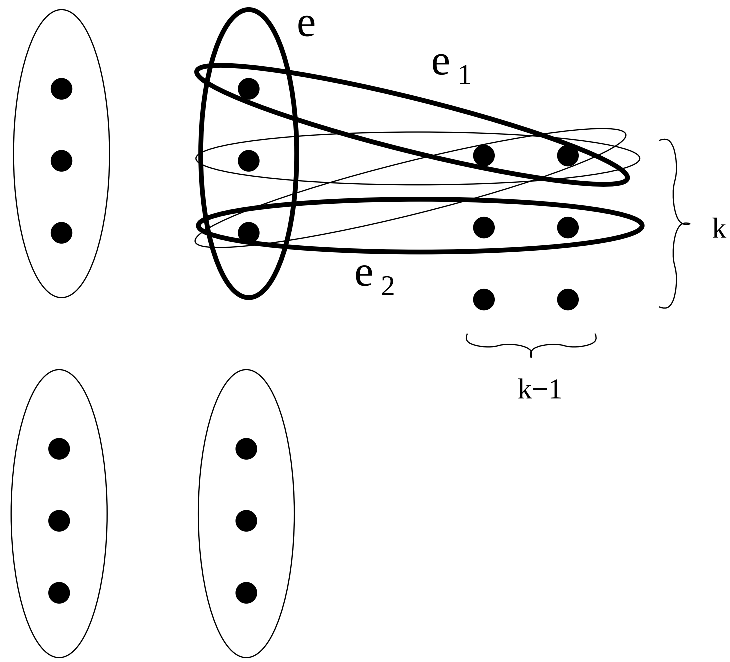

Figure 2: The proof of Proposition 2.1: the edge e is replaced by e_1 and e_2.

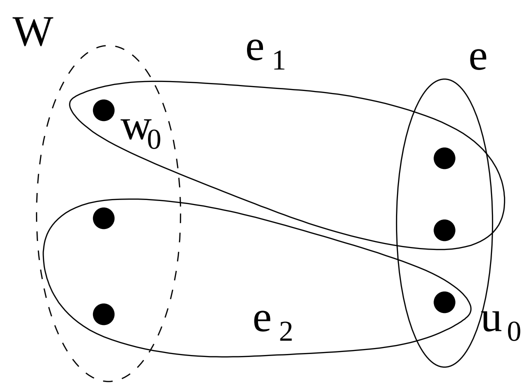

Figure 3: The edge e is friendly with respect to the set W.

The basic idea of the proof of Theorem 1.1 is to first find a relatively small, though 'powerful' matching M_0, which contains a friendly edge e_W for *every* k-tuple of vertices W. Then, we apply Proposition 2.1 (for $k = 3$) or Proposition 2.2 (for $k \geq 4$), both with $r = 1$, to the sub-hypergraph $H' = H - V(M_0)$ induced by the vertices not in M_0. This way we obtain a matching M_1 covering all vertices of H', except possibly for a set W of k vertices. Using the presence of a friendly edge e_W in M_0, the vertices in W can be "absorbed" into $M_0 \cup M_1$ to form a perfect matching of H.

In order to be able to apply Propositions 2.1 and 2.2 to the sub-hypergraph H', the 'magic' matching M_0 must be sufficiently small so that

$$\delta_{k-1}(H') \geq \delta_{k-1}(H) - |V(M_0)| \geq c(1,k)|V(H')|, \tag{7}$$

where $c(1,3) = 1/3$ and $c(1,k)$ for $k \geq 4$ is given by formula (3).

Thus, Theorem 1.1 will be proved if we show the following lemma.

Lemma 2.1 *For each $k \geq 3$ and for all $0 < c \leq 1/(10k)$, there exists $C > 0$ such that for a k-uniform n-vertex hypergraph H, where n is sufficiently large, if*

$$\delta_{k-1}(H) \geq \frac{n}{2} + C \log n,$$

then there exists a matching M_0 in H with $|V(M_0)| \leq cn$ and such that for every k-tuple of vertices W there is an edge in M_0 which is friendly with respect to W.

Note that (7) holds, and thus the proof of Theorem 1.1 follows, because we have $1/2 - c > c(1,k)$.

Our proof of Lemma 2.1 combines the probabilistic method (random matchings) with bounds on permanents (Minc conjecture). This approach was employed also in [7], and earlier, but in a different context, in [1, 9].

2.3 The proof of Lemma 2.1

We first present the idea of the proof. Observe that given W, for every $(k-2)$–element set of vertices U which is disjoint from W, there are more than $n/2$ vertices v such that for at least $2C \log n$ further vertices u, the k-tuple $e = U \cup \{v, u\}$ forms an edge of H which is friendly with respect to W (this is better explained in the proof of Claim 2.1 below).

In order to make use of this observation, we find it convenient to partition the entire vertex set $V = V(H)$ into three sets V_1, V_2, V_3, in proportion $(k -$

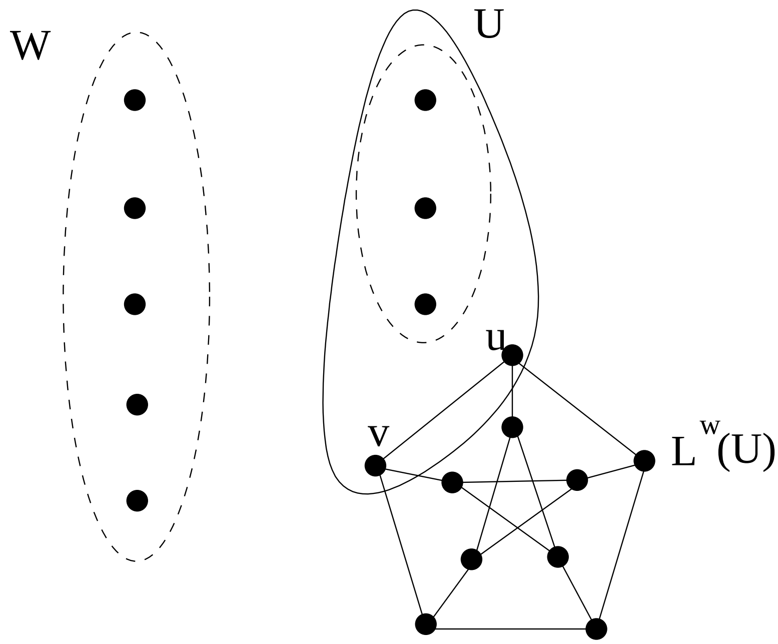

Figure 4: An illustration of Definition 2.2.

2) : 1 : 1, and will build the desired matching M_0 in three steps, corresponding to the above described ingredients of a friendly edge: $U \subset V_1$, $v \in V_2$ and $u \in V_3$ (see Figure 6).

The existence of a suitable partition V_1, V_2, V_3 (see Claim 2.2 below), as well as the choices of $v's$ and $u's$ (Claims 2.3 and 2.4) will be obtained by the probabilistic method, that is, we will analyze the respective random structures and prove that with positive probabilities they posses all the properties we need.

Definition 2.2 For each $W \in \binom{V}{k}$, let F^W be the sub-hypergraph of H consisting of all edges of H which are friendly to W. For each $U \in \binom{V \setminus W}{k-2}$ define the graph

$$L^W(U) = (V_U^W, E_U^W),$$

where E_U^W is the set of all pairs $\{v, u\}$ such that $U \cup \{v, u\} \in F^W$ and $V_U^W = \bigcup_{e \in E_U^W} e$ (see Figure 4).

Claim 2.1 *For each* $W \in \binom{V}{k}$ *and* $U \in \binom{V\setminus W}{k-2}$ *we have*

$$|V_U^W| > \frac{n}{2}$$

and

$$\delta(L^W(U)) \geq 2C \log n - 1.$$

Proof: We fix $w_0 \in W$ and will only consider edges which are friendly to W with this fixed choice of w_0. There are at least

$$\delta_{k-1} - k + 1 \geq n/2 + C \log n - k + 1 > n/2$$

choices of a vertex $v \notin W$ such that $e_1 = U \cup \{v\} \cup \{w_0\} \in H$. Given v, there are at least $2C \log n - 1$ vertices $u \neq w_0$ such that $e = U \cup \{v, u\} \in H$ and $e_2 = W \setminus \{w_0\} \cup \{u\} \in H$. Indeed, u must belong to the intersection of three sets: N_1 – the neighborhood of $U \cup \{v\}$, N_2 – the neighborhood of $W \setminus \{w_0\}$ and $V' = V \setminus (W \cup U \cup \{v\})$. Since for $i = 1, 2$

$$|N_i \cap V'| \geq \frac{n}{2} + C \log n - k,$$

there are at least

$$n + 2C \log n - 2k - (n - 2k + 1) = 2C \log n - 1$$

such vertices. Hence, each choice of v and u as above yields a friendly (with respect to W) edge e. In particular, $v \in V_U^W$, proving the first inequality of Claim 2.1, and each such v has at least $2C \log n - 1$ neighbors u in $L^W(U)$, proving the second inequality. ■

Next, we will find a suitable partition of V in such a way that the estimates of Claim 2.1 are proportionally preserved for a sub-hypergraph consisting only of the edges "spanned" by the partition. Recall that $N_G(v)$ stands for the neighborhood of a vertex v in a graph G.

Claim 2.2 *There exists a partition*

$$V = V_1 \cup V_2 \cup V_3, \quad \textit{where} \quad |V_2| = |V_3| = n/k,$$

such that for each $W \in \binom{V}{k}$ *and* $U \in \binom{V\setminus W}{k-2}$

$$|V_U^W \cap V_2| \geq \frac{n}{3k},$$

and for all $v \in V_U^W$

$$|N_{L^W(U)}(v) \cap V_3| \geq \frac{C}{k} \log n.$$

Proof: Take a random partition

$$V = V_1 \cup V_2 \cup V_3, \qquad \text{where} \qquad |V_2| = |V_3| = n/k.$$

By Claim 2.1, for all $W \in \binom{V}{k}$ and $U \in \binom{V \setminus W}{k-2}$, the expected size of $|V_U^W \cap V_2|$ is $|V_U^W|/k > n/(2k)$, and, for each $v \in V_U^W$, the expected size of $|N_{L^W(U)}(v) \cap V_3|$ is at least $(2C/k)\log n - 1/k$.

Thus, by the Chernoff bound for hypergeometric distributions (see, e.g., Thm. 2.10, inequality (2.6) in [5]),

$$\mathbb{P}(|V_U^W \cap V_2| < n/3k) = e^{-\Omega(n)}$$

and

$$\mathbb{P}(|N_{L^W(U)}(v) \cap V_3| < (C/k)\log n) \leq \exp\left\{-\frac{C}{5k}\log n\right\} = o(n^{-2k+1}),$$

provided $C \geq 10k^2$. Consequently, with probability $1 - o(1)$, for all $W \in \binom{V}{k}$, $U \in \binom{V \setminus W}{k-2}$, and all $v \in V_U^W$, both claimed inequalities hold. Thus, there exists such a partition. ■

Let us fix one partition $V = V_1 \cup V_2 \cup V_3$ guaranteed by Claim 2.2 and take an arbitrary family $\mathcal{U} = \{U_1, \ldots, U_{cn}\}$, of disjoint $(k-2)$-element subsets of V_1 (we assume for simplicity that cn is an integer). We will select a desired matching M in two random steps, involving, in turn, the sets V_2 and V_3.

Let $K(\mathcal{U}, V_2)$ be the complete bipartite graph with bipartition $(\mathcal{U}, V_2)$, and for each $W \in \binom{V}{k}$ let G_{12}^W be the graph of those pairs (U_i, v) for which $v \in V_{U_i}^W$.

Claim 2.3 *There is a subset of indices* $I \subseteq \{1, 2, \ldots, cn\}$ *of size* $|I| \geq 0.9cn$, *and there is a matching*

$$M_{12} = \{(U_i, v_i) : i \in I\}$$

in $K(\mathcal{U}, V_2)$ *such that for each* $W \in \binom{V}{k}$,

$$|M_{12} \cap G_{12}^W| \geq 0.15cn.$$

Proof: Take a random sequence $(v_1, \ldots, v_{cn})$ of the vertices from V_2, chosen one by one, uniformly at random, *with repetitions* (this corresponds to letting each U_i choose its match at random with no regard to other choices).

Let, for each $W \in \binom{V}{k}$,

$$I^W := \{i : v_i \in V_{U_i}^W\}.$$

Fact 2.1 *(i) With probability at least 1/2, the number of repetitions among $(v_1, \ldots, v_{cn})$ is at most kc^2n.*

(ii) For each $W \in \binom{V}{k}$,

$$\mathbb{P}(|I^W| < cn/4) = e^{-\Omega(n)}.$$

The proofs of all Facts will be deferred to Section 2.4. By Fact 2.1, there is a choice of $v_1, \ldots, v_{cn}$ such that for each $W \in \binom{V}{k}$ we have $|I^W| \geq cn/4$, and, at the same time, there are at least $(c - kc^2)n$ mutually distinct vertices among $v_1, \ldots, v_{cn}$. Let I be the set of indices of these distinct vertices. Then

$$|\{i \in I : v_i \in V_{U_i}^W\}| = |I^W \cap I| \geq \left(\frac{c}{4} - kc^2\right) n \geq 0.15cn,$$

where the last inequality follows from the bound $c \leq 1/(10k)$. The pairs (U_i, v_i), $i \in I$, determine a matching M_{12} in $K(\mathcal{U}, V_2)$ such that, by the definition of the graph G_{12}^W, for each $W \in \binom{V}{k}$,

$$|M_{12} \cap G_{12}^W| \geq 0.15cn.$$

Finally, note that, again by our bound on c, we have

$$|I| \geq (c - kc^2)n \geq 0.9cn.$$

This completes the proof of Claim 2.3. ■

Let M_{12} be a matching guaranteed by Claim 2.3,

$$V_2^* := V(M_{12}) \cap V_2 = \{v_i : i \in I\}.$$

For each $W \in \binom{V}{k}$, let

$$T^W := \{v_i \in V_2^* : v_i \in V_{U_i}^W\} = V(M_{12} \cap G_{12}^W) \cap V_2,$$

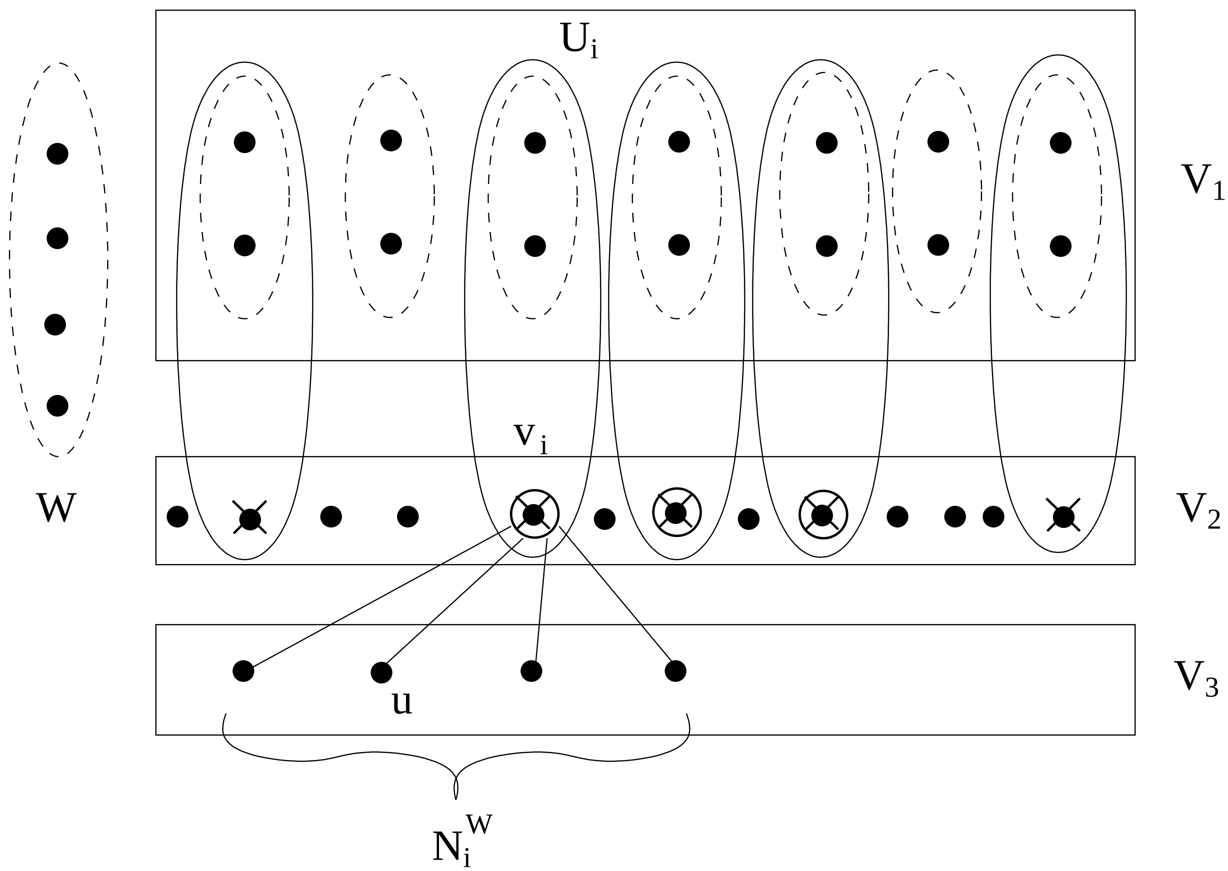

Figure 5: $T^W \subset V_2^* \subset V_2$, the elements of T^W (V_2^*) are encircled (crossed); N_i^W is the neighborhood of v_i in G_{23}^W.

and for each $v_i \in T^W$, let

$$N_i^W = \{u \in V_3 : U_i \cup \{v_i, u\} \in F^W\} = N_{L^W(U_i)}(v_i) \cap V_3,$$

where the hypergraph of friendly edges F^W and the graph L_U^W are defined in Definition 2.2 (see Figure 5). Note that by Claim 2.3,

$$|T^W| \geq 0.15cn.$$

Further, let G_{23}^W be the bipartite graph of all pairs $\{v_i, u\}$, where $v_i \in T^W$ and $u \in N_i^W$. Note that the neighborhood of each v_i in G_{23}^W is precisely the set N_i^W, and that, by Claim 2.2,

$$|N_i^W| \geq (C/k) \log n.$$

Finally, we will select a suitable V_2^*-saturating matching M_{23} in the complete bipartite graph $K(V_2^*, V_3)$.

Claim 2.4 *There is a matching*

$$M_{23} = \{(v_i, u_i) : i \in I\}$$

in $K(V_2^*, V_3)$ *such that for each* $W \in \binom{V}{k}$,

$$M_{23} \cap G_{23}^W \neq \emptyset.$$

Proof: Set $l := |V_2^*| = |I|$ and consider a random sequence $(u_1, \ldots, u_l)$ of *distinct* vertices from V_3, which can be naturally identified with the random matching M_{23}. We shall prove that for each W

$$\mathbb{P}(M_{23} \cap G_{23}^W = \emptyset) = o(n^{-k}),$$

which is sufficient to claim that there is one matching M_{23} good for *all* W's at once.

For the sake of the proof, given W, we will focus only on the sub-matching M_{23}^W saturating the subset T^W. We split the selection of M_{23}^W into two random steps. First, we choose a random subset $R \in \binom{V_3}{t}$, where $t = |T^W|$, and then we will select a random *perfect* matching in $K(T^W, R)$.

Let $\mathcal{E}_1$ be the event that for all $v_i \in T^W$ we have

$$|R \cap N_i^W| \geq 0.1cC \log n.$$

Fact 2.2

$$\mathbb{P}(\neg\mathcal{E}_1) = o(n^{-k}).$$

Let G_R be the subgraph of G_{23}^W induced by $T^W \cup R$, and let $\mathcal{E}_2$ be the event that the random perfect matching M_R in $K(T^W, R)$ satisfies

$$M_R \cap G_R \neq \emptyset.$$

Our last task will be to estimate $\mathbb{P}(\neg\mathcal{E}_2 \mid \mathcal{E}_1)$.

Fact 2.3

$$\mathbb{P}(\neg\mathcal{E}_2 \mid \mathcal{E}_1) = o(n^{-k}).$$

To quickly complete the proof of Claim 2.4, just note that by the law of total probability and by Facts 2.2 and 2.3,

$$\mathbb{P}(M_{23}^W \cap G_{23}^W = \emptyset) \leq \mathbb{P}(\neg\mathcal{E}_2 \mid \mathcal{E}_1) + \mathbb{P}(\neg\mathcal{E}_1) = o(n^{-k}).$$

■

Only now we may finish off the proof of Lemma 2.1 and thus complete the proof of Theorem 1.1 as explained in Section 2.2. Indeed, by Claim 2.4, there is a V_2^*-saturating matching M_{23} in $K(V_2^*, V_3)$ which satisfies

$$M_{23} \cap G_{23}^W \neq \emptyset$$

for all $W \in \binom{V}{k}$. This matching, together with the previously selected matching M_{12} in $K(\mathcal{U}, V_2^*)$, forms the required matching

$$M_0 = \{U_i \cup \{v_i, u_i\} : i \in I\}$$

in the hypergraph H (see Figure 6). Indeed, it follows from Claims 2.3 and 2.4 that for every $W \in \binom{V}{k}$ the matching M_0 contains a friendly edge with respect to W.

2.4 Proofs of Facts

In this section we give proofs of the three facts we used in the proof of Lemma 2.1.
Proof of Fact 2.1:

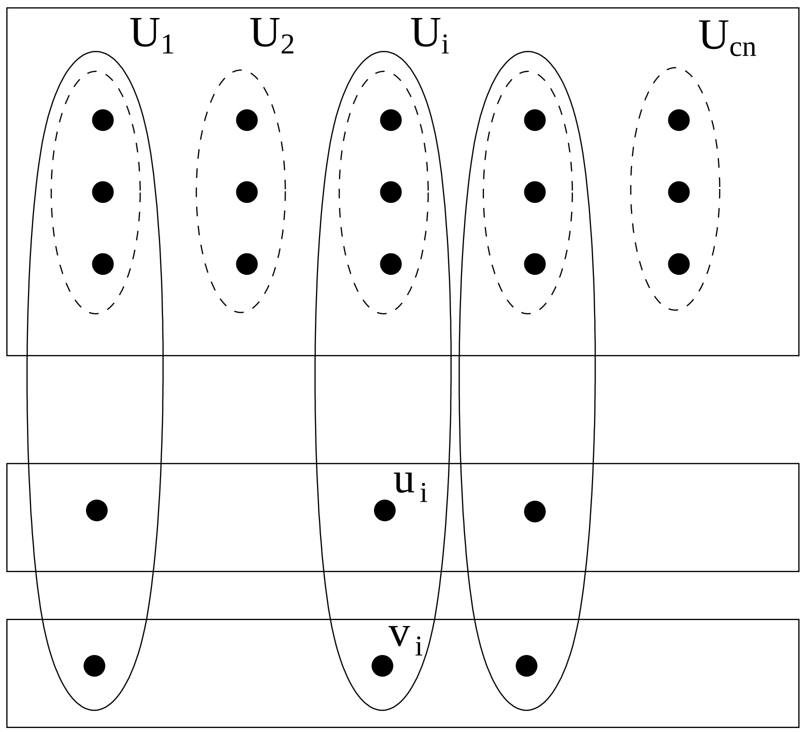

Figure 6: Matching M_0 constructed in the proof of Lemma 2.1.

(i) The expected number of repeated choices among $v_1, \ldots, v_{cn}$ is at most

$$\frac{1 + \cdots + (cn-1)}{n/k} = \frac{\binom{cn}{2}}{n/k} < \frac{k}{2}c^2 n,$$

and part *(i)* follows by Markov's inequality.

(ii) For each i such that $W \cap U_i = \emptyset$ (there are at least $cn - k$ such indices), let X_i^W be the indicator of the event that $v_i \in V_{U_i}^W$. The X_i^W's are independent, and by Claim 2.2 we have

$$\mathbb{P}(X_i^W = 1) = \frac{|V_{U_i}^W \cap V_2|}{n/k} \geq \frac{1}{3}.$$

Set $X^W = |I^W|$ and notice that $X^W = \sum_i X_i^W$ and $(EX^W) \geq (cn-k)/3 > 0.3cn$, say. Hence, part *(ii)* follows by the Chernoff bound for generalized binomial distributions (see, e.g., Thm. 2.8, inequality (2.6) in [5]). ■

Proof of Fact 2.2: For each $v_i \in T^W$, let

$$Y_i = |R \cap N_i^W|$$

As Y_i's have hypergeometric distributions with expectations

$$\frac{t|N_i^W|}{n/k} \geq \frac{(0.15cn)(C/k)\log n}{n/k} = 0.15cC\log n,$$

we have, again by the Chernoff bound,

$$\mathbb{P}(Y_i < 0.1cC\log n) = o(n^{-k-1}), \tag{8}$$

for C sufficiently large with respect to both, k and c. By (8), we have

$$\mathbb{P}(\neg\mathcal{E}) \leq \sum_{v_i \in T^W} P(Y_i < 0.1cC\log n) = o(n^{-k}).$$

■

For the proof of Fact 2.3 we will need a general result about a likely intersection of a bipartite graph with a random perfect matching of the corresponding complete bipartite graph.

Proposition 2.3 *Let A, B be two disjoint sets, $|A| = |B| = m$, and let G be a bipartite graph with the bipartition $V(G) = A \cup B$ and with dm edges for some $0 \le d = d(m) \le m$. Further, let M be a random perfect matching in the complete bipartite graph $K(A, B)$. Then*

$$\mathbb{P}(M \cap G = \emptyset) = O\left(e^{-d/4}\right).$$

We defer the proof to the end of this section.

Proof of Fact 2.3: Recall that G_R is a bipartite graph with bipartition (T^W, R), where $|T^W| = |R| = t$, and that M_R is a random perfect matching in the complete bipartite graph $K(T^W, R)$. We are to show that

$$\mathbb{P}(M_R \cap G_R = \emptyset \mid \mathcal{E}_1) = o(n^{-k}). \tag{9}$$

Note that, by conditioning on $\mathcal{E}_1$, each vertex of T^W has degree at least $d := 0.1cC \log n$ in G_R. Consequently, $|E(G_R)| \ge dm$, and Proposition 2.3 yields (9) for sufficiently large C. ∎

Proof of Proposition 2.3: Without loss of generality we may assume that $d \le m/2$, since otherwise we could take a subgraph G' of G with $e(G') = \lfloor m^2/2 \rfloor$, noticing that

$$\mathbb{P}(M \cap G = \emptyset) \le \mathbb{P}(M \cap G' = \emptyset).$$

Because of this initial adjustment, in order to prove Proposition 2.3, we will have to show that

$$\mathbb{P}(M \cap G = \emptyset) = O\left(e^{-d/2}\right).$$

If $M \cap G = \emptyset$, then $M \subseteq \overline{G}$, where $\overline{G} = K(A, B) - G$ is the bipartite complement of G. Thus

$$\mathbb{P}(M \cap G = \emptyset) = \mathbb{P}(M \subseteq \overline{G}) = M(\overline{G})/m!, \tag{10}$$

where $M(\overline{G})$ is the number of perfect matchings in $\overline{G}$. Let J be the adjacency matrix of $\overline{G}$. Then $M(\overline{G})$ is equal to the permanent of J.

Let $\bar{d}_1, \ldots, \bar{d}_m$ be the degrees of the vertices from A in $\overline{G}$, which at the same time are the row totals of J. Note that

$$\sum_i \bar{d}_i = (m - d)m.$$

and that we may assume that $\delta(\overline{G}) \geq 1$, since otherwise $\mathbb{P}(M \subseteq \overline{G}) = 0$. Using Brégman's celebrated upper bound on the permanent (known also as the Minc Conjecture), see [2] for a probabilistic proof, we infer that

$$M(\overline{G}) \leq \prod_{i=1}^{m} \bar{d}_i!^{1/\bar{d}_i}. \tag{11}$$

One can check that the above quantity is maximized when all $\bar{d}_i$'s are as close to the average $\bar{d} = m - d$ as possible. Indeed, it is enough to verify, for all integers $x \geq 1$, the inequality

$$x!^{1/x}(x+2)!^{1/(x+2)} \leq (x+1)!^{2/(x+1)}$$

or, equivalently,

$$\left(\frac{x+2}{x+1}\right)^{x^2+x} \leq \frac{(x+1)^{2x}}{x!^2}.$$

The LHS of the latter inequality is, clearly, smaller than e^x, On the RHS we use Stirling's bound $x! < e^{1/12x}\sqrt{2\pi x}(x/e)^x$, and check that the resulting quantity is larger than e^x (for $x \geq 4$ it follows from the inequality $e^x > 7x$, while for $x = 1, 2, 3$ we just plug in the numbers.)

Assuming for clarity of exposition that $\bar{d}$ is an integer, we thus have

$$\prod_{i=1}^{m} \bar{d}_i!^{1/\bar{d}_i} \leq \left(\bar{d}!^{1/\bar{d}}\right)^m. \tag{12}$$

Now we need to refer again to Stirling's estimates of the factorials. In a weaker form they yield for each x and some $c_1, c_2 > 0$,

$$c_1\sqrt{x}(x/e)^x < x! < c_2\sqrt{x}(x/e)^x.$$

So, using also the bound $m/d \leq 2$, we have

$$\frac{1}{m!}\left(\bar{d}!^{1/\bar{d}}\right)^m < \frac{1}{c_1\sqrt{m}}(c_2\sqrt{\bar{d}})^{m/\bar{d}}\left(\frac{e}{m}\right)^m\left(\frac{\bar{d}}{e}\right)^m = O\left(\sqrt{\frac{\bar{d}^{m/\bar{d}}}{m}}\left(\frac{\bar{d}}{m}\right)^m\right). \tag{13}$$

However, it can be easily checked that

$$\frac{\bar{d}^{m/\bar{d}}}{m} \leq \left(\frac{m}{\bar{d}}\right)^m, \tag{14}$$

and hence, by (10–14)

$$\frac{M(\overline{G})}{n!} = O\left(\left(\frac{\bar{d}}{m}\right)^{m/2}\right) = O\left(\left(1-\frac{d}{m}\right)^{m/2}\right) = O\left(e^{-d/2}\right).$$

■

3 Fractional perfect matchings

The well-known Farkas Lemma (see, e.g., [3] or [8]) asserts that the system $\mathbf{Ax} \le \mathbf{0}, \mathbf{bx} > 0$ is unsolvable if and only if the system $\mathbf{yA} = \mathbf{b}, \mathbf{y} \ge \mathbf{0}$ is solvable. Using this classic result we will now show a degree condition for the existence of a fractional perfect matching in a k-uniform hypergraph. As graphs with fractional perfect matchings are fully characterized by a Hall -type condition (see, e.g., [8]), we from now on assume that $k \ge 3$.

Let $\Delta_{k-1}(H)$ be the maximum of $|N_H(v_1,\dots,v_{k-1})|$ over all $(k-1)$-tuples of vertices in H, and let G_H be the $(k-1)$-uniform hypergraph of all $(k-1)$-tuples of vertices with $|N_H(v_1,\dots,v_{k-1})| < n/k$.

It turns out that a fractional perfect matching is guaranteed even if we allow several $(k-1)$-tuples of vertices to have their degree smaller than n/k (even zero), provided they are not clustered too much. The next result is in a sense optimal.

Proposition 3.1 *If $|V(H)| = n$ and $\Delta_{k-2}(G_H) \le (k-2)(n/k-1)$ then H has a fractional perfect matching. Moreover, there exists an n-vertex k-uniform hypergraph with $\Delta_{k-2}(G_H) > (k-2)(n/k-1)$ having no fractional perfect matching.*

Proof: We apply Farkas' Lemma with $\mathbf{A}$ – the incidence matrix of H and $\mathbf{b}$ – the vector of length n whose all entries are equal to 1. All we need is to show that the system of inequalities $\mathbf{Ax} \le \mathbf{0}, \mathbf{bx} > 0$ has no solutions. Suppose that $x_1,\dots,x_n$ is a solution to the system $\mathbf{Ax} \le \mathbf{0}$. We will show that $\mathbf{bx} \le 0$.

Let us identify the vertices of H with the values $x_1,\dots,x_n$ assigned to them, and without loss of generality assume that $x_1 \ge x_2 \ge \cdots \ge x_n$. Let s be the smallest index for which $|N_H(x_1,\dots,x_{k-2},x_s)| \ge n/k$. By our assumption,

$$s \le (k-2)\left(\frac{n}{k}-1\right) + k - 1 = n - \frac{2n}{k} + 1.$$

For the sake of clarity, assume first that n is divisible by k.

Let $Z \subset N_H(x_1, \ldots, x_{k-2}, x_s)$, $|Z| = n/k$. Then, because $\mathbf{Ax} \leq \mathbf{0}$, we have

$$z + x_s + x_1 + \cdots + x_{k-2} \leq 0 \tag{15}$$

for each $z \in Z$. Let us partition all vertices of H into disjoint sets T_i, $i = 1, \ldots, n/k$, of size k, so that each set T_i consists of one vertex $z^{(i)} \in Z$ and one vertex $y^{(i)} \leq x_s$, while the remaining $k-2$ vertices can be arbitrary. Owing to the upper bound on s, there are at least $2n/k$ vertices $x_j \leq x_s$, and so, such a partition always exists.

Note that for each i, by (15), we have

$$\sum_{x \in T_i} x \leq \max_{z \in Z} z + x_s + x_1 + \cdots + x_{k-2} \leq 0, \tag{16}$$

which implies that $\sum_{i=1}^n x_i \leq 0$, that is, $\mathbf{bx} \leq 0$.

In the general case, when n is not necessarily divisible by k, we will estimate $k \sum_{i=1}^n x_i$ instead. More specifically, we will find sets $T^{(1)}, \ldots, T^{(n)}$ of size k so that each vertex is contained in precisely k of them. To achieve this goal, we "clone" each x_j into k elements $x_j^{(l)}$, $l = 1, \ldots, k$, where

$$x_j^{(1)} = \cdots = x_j^{(k)} = x_j.$$

for each $j = 1, \ldots, n$. (Remember that we have identified each vertex x_j with the weight assigned to it.)

Also, for each $l = 1, \ldots, k$, we choose a subset

$$Z^{(l)} = \{x_{j_1}^{(l)}, \ldots, x_{j_{m^{(l)}}}^{(l)}\} \subset \{x_1^{(l)}, \ldots, x_n^{(l)}\}$$

such that, as before,

$$\{x_{j_1}, \ldots, x_{j_{m^{(l)}}}\} \subset N_H(x_1, \ldots, x_{k-2}, x_s)$$

and

$$\sum_{l=1}^k |Z^{(l)}| = \sum_{l=1}^k m^{(l)} = n.$$

(This is always possible, since $d_H(x_1, \ldots, x_{k-2}, x_s) \geq n/k$.)

Now, we partition all the kn elements $x_j^{(l)}$, $l = 1, \ldots, k$, $j = 1, \ldots, n$, into n disjoint sets T_i of size k, so that, as before, each of them contains one

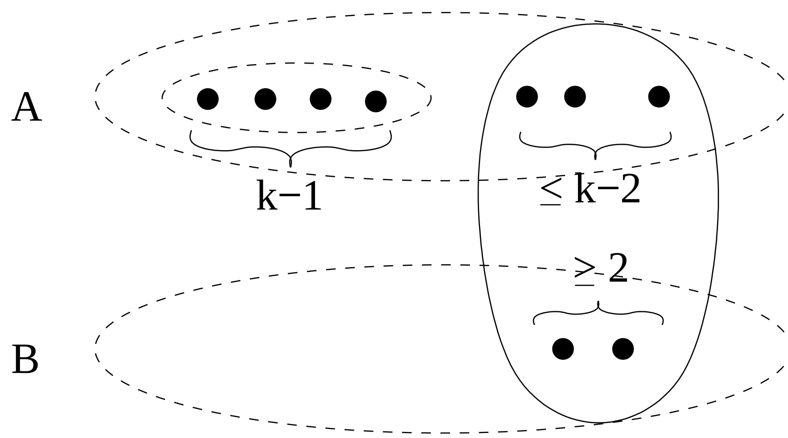

Figure 7: An extremal hypergraph without a fractional perfect matching used in the proof of the second part of Proposition 3.1.

element $z^{(i)} \in \bigcup_{l=1}^{k} Z^{(l)}$, and one element $y^{(i)} \le x_s$. Since there are at least $k\lceil 2n/k\rceil \ge 2n$ elements $x_j^{(l)} \le x_s$, this is always possible.

Finally, since (16) holds for each $i = 1, \ldots, n$, we have

$$k\sum_{j=1}^{n} x_j = \sum_{l=1}^{k}\sum_{j=1}^{n} x_j^{(l)} = \sum_{i=1}^{n}\sum_{x\in T_i} x \le 0.$$

To prove the second part of Proposition 3.1, take two disjoint sets, A and B, where

$$|A| = \lfloor (k-2)n/k\rfloor + 1 \qquad \text{and} \qquad |B| = n - |A|,$$

and construct a k-uniform hypergraph H_0 with the vertex set $V(H_0) = A\cup B$ and the edge set consisting of all k-tuples with at least two vertices in B (see Figure 7; this example was found by J. Polcyn). The only $(k-1)$-tuples of degree less than n/k (in fact, of degree 0) are those contained in A. Thus,

$$\Delta_{k-2}(G_{H_0}) = |A| - (k-2) = \lfloor (k-2)n/k\rfloor - (k-2) + 1 > (k-2)(n/k-1).$$

Suppose there is a fractional perfect matching in H_0. Then the total weight of the edges of H_0 is at least $|A|/(k-2) > n/k$, a contradiction with (2). ■

As an immediate corollary we obtain the following degree threshold result for fractional perfect matchings in k-uniform hypergraphs. For all integers $k \geq 3$, denote by $t_k^*(n)$ the smallest integer t such that every k-uniform hypergraph on n vertices and with $\delta_{k-1} \geq t$ has a perfect fractional matching.

Corollary 3.1 *For all $k \geq 3$ we have*

$$t_k^*(n) = \lceil n/k \rceil.$$

Proof: To prove that $t_k^*(n) \leq \lceil n/k \rceil$, let H be an arbitrary k-uniform, n-vertex hypergraph with $\delta_{k-1} \geq \lceil n/k \rceil$. Then, $G_H = \emptyset$, and the assumption of Proposition 3.1 is vacuously satisfied. Hence, H has a fractional perfect matching..

For the lower bound on $t_k^*(n)$, take two disjoint sets, A and B, where

$$|A| = n - \lceil n/k \rceil + 1 > n - n/k \qquad \text{and} \qquad |B| = \lceil n/k \rceil - 1.$$

Construct a k-uniform hypergraph H_1 with vertex set $V(H_1) = A \cup B$ and edge set consisting of all k-tuples with at least one vertex in B. Note that $\delta_{k-1}(H_1) = |B|$. On the other hand, if there was a fractional perfect matching in H_1, then the total weight of all the edges would be at least $|A|/(k-1) > n/k$, a contradiction with (2).

References

[1] N. Alon, V. Rödl and A. Ruciński, Perfect matchings in ϵ-regular graphs, *The Electr. J. of Combin.* 5(1) (1998) #R13.

[2] N. Alon and J. Spencer, *The Probabilistic Method*, 2nd Ed., John Wiley and Sons, New York (2000).

[3] V. Chvátal, *Linear Programming*, W. H. Freeman, New York (1983).

[4] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford Ser.* 12 (1961) 313-320.

[5] S. Janson, T. Łuczak and A. Ruciński, *Random Graphs*, John Wiley and Sons, New York (2000).

[6] G. Y. Katona and H. A. Kierstead, Hamiltonian chains in hypergraphs, *J. Graph Theory* 30 (1999) 205-212.

[7] D. Kühn and D. Osthus, Matchings in hypergraphs of large minimum degree, *J. Graph Theory*, to appear.

[8] L. Lovász & M.D. Plummer, *Matching theory.* North-Holland Mathematics Studies 121, Annals of Discrete Mathematics 29, North-Holland Publishing Co., Amsterdam; Akadémiai Kiadó, Budapest, 1986.

[9] V. Rödl and A. Ruciński, Perfect matching in ϵ-regular graphs and the Blow-up Lemma, *Combinatorica* 19 (3) (1999) 437-452.

[10] V. Rödl, A. Ruciński, and E. Szemeredi, A Dirac-type theorem for 3-uniform hypergraphs, *Combinatorics, Probability, and Computing*, to appear.

[11] V. Rödl, A. Ruciński, and E. Szeméredi, Dirac theorem for 3-uniform hypergraphs, work in progress.

[12] V. Rödl, A. Ruciński, and E. Szeméredi, Perfect matchings in uniform hypergraphs with large minimum degree. II, work in progress.

[13] V. Rödl, A. Ruciński, and E. Szeméredi, An approximative Dirac-type theorem for k-uniform hypergraphs, submitted.