Rainbow Hamilton cycles in uniform hypergraphs

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Abstract

Let $K_n^{(k)}$ be a complete $k$-uniform hypergraph. A loose Hamilton cycle in $K_n^{(k)}$ is a cycle of order $n$ in which every pair of adjacent edges intersects in a single vertex. We color every edge of $K_n^{(k)}$ in such way that no color appears more than $cn^{k-1}$ times for some constant $c = c(k)$. We show that if $n$ is sufficiently large then there is a loose Hamilton cycle in which each edge is a different color. We also prove a similar result for colorings for which the maximum degree of the hypergraph induced by any single color is bounded by $c'n^{k-1}$ for some constant $c' = c'(k)$. In this case we show that there exists a properly colored loose Hamilton cycle. Both results are best possible up to the constants. We also determine results for other types of Hamilton cycles.

1 Introduction

By a coloring of a hypergraph $H$ we mean any function $\phi : H \to \mathbb{N}$ assigning natural numbers (colors) to the edges of $H$. (In this paper we do not consider vertex colorings.) A hypergraph $H$ together with a given coloring $\phi$ will be dubbed a colored hypergraph. A subhypergraph $F$ of a colored hypergraph $H$ is said to be properly colored if every two adjacent edges of $F$ receive different colors. (Two different edges are adjacent if they share at least one vertex.) We say that a subhypergraph $F$ of a colored hypergraph $H$ is rainbow if every edge of $F$ receives a different color, that is, when $\phi$ is injective on $F$.

In order to force the presence of properly colored or rainbow subhypergraphs one has to restrict the colorings $\phi$, either globally or locally. A coloring $\phi$ is $r$-bounded if every color

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is used at most \( r \) times, that is, \( |\phi^{-1}(i)| \leq r \) for all \( i \in \mathbb{N} \). A coloring \( \phi \) is \( r \)-degree bounded if the hypergraph induced by any single color has maximum degree bounded by \( r \), that is, \( \Delta(H[\phi^{-1}(i)]) \leq r \) for all \( i \in \mathbb{N} \).

In this paper we study the existence of properly colored and rainbow Hamilton cycles in colored \( k \)-uniform complete hypergraphs, \( k \geq 3 \). (A hypergraph is \( k \)-uniform if every edge has size \( k \); it is complete if all \( k \)-element subsets of the vertices form edges.) There is a broad literature on this subject for \( k = 2 \), that is, for graphs. Indeed, setting \( r = cn \), Alon and Gutin proved in [2], improving upon earlier results from [5, 6, 16] that if \( c < 1 - 1/\sqrt{2} \) then any \( r \)-degree bounded coloring of the edges of the complete graph \( K_n \) yields a properly colored Hamilton cycle (for the history of the problem, see [3]). It had been conjectured in [5] that the constant \( 1 - 1/\sqrt{2} \) can be replaced by \( 1/2 \) which is the best possible. Rainbow Hamilton cycles in \( r \)-bounded colorings of the complete graph have been studied in [1, 8, 10, 12]. Hahn and Thomassen conjectured that their existence is guaranteed if \( r = cn \) for some \( c > 0 \). This was confirmed by Albert, Frieze, and Reed in [1] with \( c = 1/64 \). Again, \( c = 1/2 \) seems to be a critical value here, since one can use each of \( n - 1 \) colors exactly \( n/2 \) times, making the presence of rainbow Hamilton cycles impossible. In striking contrast, there is literally nothing known on properly colored or rainbow Hamilton cycles in colored \( k \)-uniform hypergraphs for \( k \geq 3 \).

The notion of a hypergraph cycle can be ambiguous. In this paper we are not concerned with the Berge cycles as defined by Berge in [4] (see also [11]). Instead, following a recent trend in the literature ([7, 13, 15]), given an integer \( 1 \leq \ell < k \), we define an \( \ell \)-overlapping cycle as a \( k \)-uniform hypergraph in which, for some cyclic ordering of its vertices, every edge consists of \( k \) consecutive vertices, and every two consecutive edges (in the natural ordering of the edges induced by the ordering of the vertices) share exactly \( \ell \) vertices. (See Fig. 1 for an example of a 2-overlapping and a 3-overlapping 5-uniform cycle.)

The two extreme cases of \( \ell = 1 \) and \( \ell = k - 1 \) are referred to as, respectively, loose and tight cycles. Note that the number of edges of an \( \ell \)-overlapping cycle with \( s \) vertices is \( s/(k - \ell) \). Note also that when \( k - \ell \) divides \( s \), every tight cycle on \( s \) vertices contains an \( \ell \)-overlapping cycle on the same vertex set (with the same cyclic ordering).

Given a \( k \)-uniform hypergraph \( H \) on \( n \) vertices, where \( k - \ell \) divides \( n \), an \( \ell \)-overlapping cycle contained in \( H \) is called Hamilton if it goes through every vertex of \( H \), that is, if \( s = n \). We denote such a Hamilton cycle by \( H_n^\ell(k) \).

In this paper we prove the following two results. Let \( K_n^{(k)} \) be the complete \( k \)-uniform hypergraph of order \( n \).
Theorem 1.1 For every $1 \leq \ell < k$ there is a constant $c = c(k, \ell)$ such that if $n$ is sufficiently large and $k - \ell$ divides $n$ then any $cn^{k-\ell}$-bounded coloring of $K_n^{(k)}$ yields a rainbow copy of $C_n^{(k)}(\ell)$.

Theorem 1.2 For every $1 \leq \ell < k$ there is a constant $c' = c'(k, \ell)$ such that if $n$ is sufficiently large and $k - \ell$ divides $n$ then any $c'n^{k-\ell}$-degree bounded coloring of $K_n^{(k)}$ yields a properly colored copy of $C_n^{(k)}(\ell)$.

Note that for loose Hamilton cycles (i.e. $\ell = 1$) the above results are optimal up to the values of $c$ and $c'$. Theorem 1.2 is trivially optimal, as the largest maximum degree can be at most $r = (n-1)/k! n^{k-1}$. To see that also Theorem 1.1 is optimal up to the constant for $\ell = 1$ consider any coloring of $K_n^{(k)}$ using each color precisely $r = \frac{n}{k!} - 1 \sim \frac{k-1}{k!} n^{k-1}$ times, and thus using only $\frac{n}{k!} - 1$ colors altogether. Such a coloring is $r$-bounded and, clearly, there is no rainbow copy of $C_n^{(k)}(1)$.

Problem 1.3 For all $k \geq 3$ and $\ell = 1$, determine sup $c$ and sup $c'$ over all values of $c$ and, respectively, $c'$ for which Theorems 1.1 and 1.2 hold.

We believe that Theorems 1.1 and 1.2 are optimal up to the constants also for $\ell \geq 2$, that is, we believe that the answer to the following question is positive.

Problem 1.4 For all $k \geq 3$ and $2 \leq \ell \leq k - 1$, does there exist an $r$-bounded ($r$-degree bounded) coloring $\phi$ of $K_n^{(k)}$ such that $r = \Theta(n^{k-\ell})$ and no copy of $C_n^{(k)}(\ell)$ is rainbow (properly colored)?

As some evidence supporting our belief, consider the bipartite version of both problems for $k = 3$ and $\ell = 2$. Let $K_{n,2n}^{(3)} = (V_1, V_2, E)$, where $|V_1| = n$, $|V_2| = 2n$ and $E = \{e \subset V_1 \cup V_2 : |e \cap V_i| = i, \ i = 1, 2\}$. To every edge $e$ assign the pair $e \cap V_2$ as its color. Clearly, every color appears exactly $n$ times and hence such a coloring is $n$-bounded (and thus $n$-degree bounded). Finally, note that every tight Hamilton cycle in $K_{n,2n}^{(3)}$ induces a cyclic sequence of vertices with a repeated pattern of two vertices from $V_2$ followed by one vertex from $V_1$. Hence, there is a pair of consecutive edges with the same color (actually, there are $n$ such pairs), and so no copy of a properly colored (or rainbow) $C_n^{(3)}(2)$ exists.

2 The proofs

We will need a special version of the Lovász Local Lemma. A similar result was already established in [1, 9, 14]. Contrary to the above results, in our formulation of the lemma we avoid conditional probabilities so that we do not need to make a priori assumptions that certain events have positive probability.
Lemma 2.1 Let $A_1, A_2, \ldots, A_m$ be events in an arbitrary probability space $\Omega$. For each $1 \leq i \leq m$, let $[m] \setminus \{i\} = X_i \cup Y_i$ be a partition of the index set $[m] \setminus \{i\}$ and let

$$d = \max\{|Y_i| : 1 \leq i \leq m\}.$$ 

If for each $1 \leq i \leq m$ and all $X \subseteq X_i$

$$\Pr \left( A_i \cap \bigcap_{j \in X} \overline{A}_j \right) \leq \frac{1}{4(d + 1)} \Pr \left( \bigcap_{j \in X} \overline{A}_j \right) \quad (1)$$

then $\Pr \left( \bigcap_{i=1}^m \overline{A}_i \right) > 0$. (We adopt the convention that $\bigcap_{j \in \emptyset} \overline{A}_j = \Omega$.)

Proof. We prove by induction on $t = 1, \ldots, m$, that for every $T \subseteq [m]$, $|T| = t$, and for every $i \in T$, setting $S = T \setminus \{i\}$, we have

$$\Pr \left( \bigcap_{j \in T} \overline{A}_j \right) > 0 \quad \text{and} \quad \Pr \left( A_i \bigcap_{i \in S} \overline{A}_j \right) \leq \frac{1}{2(d + 1)}. \quad (2)$$

For $t = 1$ we apply the (1) with $X = \emptyset$, obtaining for each $i$ that $\Pr(A_i) \leq \frac{1}{4(d + 1)}$, equivalently $\Pr(\overline{A}_i) \geq 1 - \frac{1}{4(d + 1)} > 0$, which confirms (2) for $t = 1$.

Now, assume truth for some $t$, $1 \leq t \leq m - 1$, and consider a set $T = \{i\} \cup S$, where $i \notin S$ and $|S| = t$. Set $X = S \cap X_i$ and $Y = S \cap Y_i$, and observe that $S = X \cup Y$ and $|Y| \leq |Y_i| \leq d$. By the induction assumption $\Pr(\bigcap_{j \in S} \overline{A}_j) > 0$. If $Y = \emptyset$ (and thus $X = S$), by our assumption (1),

$$\Pr \left( A_i \bigcap_{j \in S} \overline{A}_j \right) \leq \frac{1}{4(d + 1)}.$$ 

Otherwise, $|X| < |S| = t$ and, again by (1) (in the numerator) and the induction assumption (in the denominator) we argue that

$$\Pr \left( A_i \bigcap_{j \in S} \overline{A}_j \right) = \frac{\Pr \left( A_i \bigcap_{j \in S} \overline{A}_j \bigcap_{j \in X} \overline{A}_j \right)}{\Pr \left( \bigcap_{j \in Y} \overline{A}_j \bigcap_{j \in X} \overline{A}_j \right)} \leq \frac{\frac{1}{4(d + 1)}}{1 - |Y| \frac{1}{2(d + 1)}} \leq \frac{1}{2(d + 1)}.$$ 

Thus,

$$\Pr \left( \bigcap_{j \in T} \overline{A}_j \right) = \Pr \left( A_i \bigcap_{j \in S} \overline{A}_j \right) \Pr(\bigcap_{j \in S} \overline{A}_j) > 0,$$

which completes the proof of Lemma 2.1.

The proofs of Theorems 1.1 and 1.2 extend some ideas introduced by Albert, Frieze and Reed in [1] and are based on the following technical result.

Proposition 2.2 For all $1 \leq \ell < k$ there exist constants $\delta = \delta(k, \ell)$, $0 < \delta < 1$, such that for every pair $e, f$ of edges of $K_n^{(k)}$ with $|e \cap f| \leq \ell$ and for every set $X$ of pairs $g, h$ of edges of $K_n^{(k)}$ satisfying $(g \cup h) \cap (e \cup f) = \emptyset$ the following holds. Let
• \( C \) be the set of all copies \( C \) of \( C_n^{(k)}(k - 1) \) in \( K_n^{(k)} \) such that \( \{g, h\} \not\subseteq C \) for all \( \{g, h\} \in X \), and

• \( C_1 = \{C \in C : \{e, f\} \subset C\} \).

Then, if \( C_1 \neq \emptyset \), one can find a disjoint family \( \{S_C : C \in C_1\} \) of sets of copies of \( C_n^{(k)}(k - 1) \) from \( C \) (indexed by the copies \( C \in C_1 \)) such that for all \( C \in C_1 \) we have \( |S_C| \geq \delta n^{2k-\ell-1} \).

First we show that Proposition 2.2 together with Lemma 2.1 imply both Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.1.** Let \( 1 \leq \ell < k \) and set \( c = \frac{\delta}{10k} \), where \( \delta = \delta(k, \ell) \) is the constant given by Proposition 2.2. Fix a \( cn^{k-\ell} \)-bounded coloring \( \phi \) of \( K_n^{(k)} \) and define

\[
M = \{\{e, f\} : e, f \in K_n^{(k)}, |e \cap f| \leq \ell \text{ and } \phi(e) = \phi(f)\}.
\]

Moreover, for every pair \( \{e, f\} \in M \) set

\[
A_{e, f} = \{C \subset K_n^{(k)} : C \cong C_n^{(k)}(k - 1) \text{ and } \{e, f\} \subset C\}.
\]

In order to prove Theorem 1.1 it suffices to show that

\[
\bigcap_{\{e, f\} \in M} A_{e, f} \neq \emptyset. \quad (3)
\]

Indeed, if (5) is true then there is a tight Hamilton cycle \( C \cong C_n^{(k)}(k - 1) \) such that for every pair of its edges \( e \) and \( f \) with \( |e \cap f| \leq \ell \) we have \( \phi(e) \neq \phi(f) \). Since, by assumption, \( k - \ell \) divides \( n \), \( C \) contains a copy of \( C_n^{(k)}(\ell) \) which is rainbow, as required.

To prove (5) we apply the probabilistic method and Lemma 2.1. To this end, for a given pair \( \{e, f\} \in M \) let

\[
Y_{e, f} = \{\{e', f'\} \in M : \{e', f'\} \neq \{e, f\} \text{ and } (e \cup f) \cap (e' \cup f') \neq \emptyset\}
\]

and

\[
X_{e, f} = M \setminus (Y_{e, f} \cup \{e, f\}).
\]

To estimate \( d \), we bound from above the size of \( Y_{e, f} \) as follows. For given edges \( e \) and \( f \) we can find at most \( 2kn^{k-1} \) edges \( e' \) sharing a vertex from \( e \cup f \). For every such \( e' \) we have at most \( cn^{k-\ell} \) candidates for \( f' \), since \( e' \) and \( f' \) must have the same color. Thus,

\[
d = \max_{\{e, f\} \in M} |Y_{e, f}| \leq 2ckn^{2k-\ell-1} \leq \frac{\delta n^{2k-\ell-1}}{4} - 1.
\]

Now, let us consider a uniform probability space consisting of all tight Hamilton cycles \( C \cong C_n^{(k)}(k - 1) \) in \( K_n^{(k)} \). In order to prove (5), and thus finish the proof of Theorem 1.1, it suffices to show that

\[
\Pr\left( \bigcap_{\{e, f\} \in M} A_{e, f} \right) > 0. \quad (4)
\]
Thus, it remains to verify assumption (1) of Lemma 2.1. Fix \( \{e, f\} \in M \) and \( X \subseteq X_{e, f} \). If \( C_1 = \emptyset \), there is nothing to prove. Otherwise, by Proposition 2.2,

\[
\frac{\Pr \left( A_{e, f} \cap \bigcap \{e', f'\} \in X \: A_{e', f'} \right)}{\Pr \left( \bigcap \{e', f'\} \in X \: A_{e', f'} \right)} = \frac{|C_1|}{|C|} \leq \frac{|C_1|}{\sum_{C \in C_1} |S_C|} \leq \frac{1}{\delta n^{2k-\ell} - 1} \leq \frac{1}{4(d + 1)}.
\]

Hence, we are in position to apply Lemma 2.1 with \( m = |M| \), \( A_i := A_{e, f} \), \( X_i := X_{e, f} \), and \( Y_i := Y_{e, f} \), and conclude that (4), and consequently (5) holds. This completes the proof of Theorem 1.1. \( \square \)

**Proof of Theorem 1.2.** This proof goes along the lines of the proof of Theorem 1.1. Let \( c' = \frac{\delta}{10k^2} \), where \( \delta = \delta(k, \ell) \) is the constant given by Proposition 2.2. Fix a \( dn^{k-\ell} \)-degree bounded coloring of \( K_n^{(k)} \). Here we slightly modify the definition of \( M \). Let

\[
M = \{ \{e, f\} : e, f \in K_n^{(k)}, 1 \leq |e \cap f| \leq \ell \text{ and } \phi(e) = \phi(f) \}.
\]

As before,

\[
A_{e, f} = \{ C \subset K_n^{(k)} : C \cong C_n^{(k)}(k - 1) \text{ and } \{e, f\} \subset C \}.
\]

and in order to prove Theorem 1.2 it suffices to show that

\[
\bigcap_{\{e, f\} \in M} \overline{A_{e, f}} \neq \emptyset. \tag{5}
\]

Indeed, if (5) is true then there is a tight Hamilton cycle \( C \cong C_n^{(k)}(k - 1) \) such that for every pair of its edges \( e \) and \( f \) with \( 1 \leq |e \cap f| \leq \ell \) we have \( \phi(e) \neq \phi(f) \). Since, by assumption, \( k - \ell \) divides \( n \), \( C \) contains a copy of \( C_n^{(k)}(\ell) \) which is properly colored, as required.

We define sets \( Y_{e, f} \) and \( X_{e, f} \) as before and recalculate the upper bound on \( |Y_{e, f}| \). For given edges \( e \) and \( f \) we can find at most \( 2kn^{k-1} \) edges \( e' \) sharing a vertex from \( e \cup f \). For every such \( e' \) we have at most \( c'kn^{k-\ell} \) candidates for \( f' \) since \( e' \) and \( f' \) intersect and have the same color. Thus,

\[
|Y_{e, f}| \leq 2c'k^2n^{2k-\ell} \leq \frac{\delta n^{2k-\ell} - 1}{4} - 1.
\]

The rest of the proof is identical to the proof of Theorem 1.1 and therefore is omitted. \( \square \)

### 3 Proof of Proposition 2.2

Let \( e \) and \( f \) be given edges in \( K_n^{(k)} \) such that \( |e \cap f| \leq \ell \) and let \( C \in C_1 \) be a tight Hamilton cycle containing \( e \) and \( f \) and missing at least one edge from each pair \( \{g, h\} \in X \). We describe two constructions depending on the size of \( e \cap f \).

**Construction 1:** for \( 2 \leq |e \cap f| \leq \ell \).

Let \( |e \cap f| = a \) and let \( e = (u_1, \ldots, u_k) \) and \( f = (v_1, \ldots, v_k) \) be such that \( u_{k-a+1} = v_1, u_{k-a+2} = v_2, \ldots, u_k = v_a \). This way we fix an orientation of \( C \) where \( e \) precedes \( f \). Let \( P = C \setminus \{e \cup f\} \) be the segment of \( C \) between \( f \) and \( e \) of length \( n - 2k + a \). We select arbitrarily \( 2k - a - 1 \) vertex disjoint edges \( g_1, \ldots, g_{2k-a-1} \) from \( P \), so that \( C \) is of the form...
It is easy to check that every new edge intersects a vertex from \( e \cup f \). Thus, \( \tilde{C} \setminus C \) contains no edge from any pair of edges belonging to \( X \). Moreover, note that different choices of \( g_i \) yield different cycles \( \tilde{C} \). Thus, \( |S(C)| = \Omega(n^{2k-\ell-1}) \).

It remains to show that for any two tight Hamilton cycles \( C \neq C' \in C_1 \) we have \( S(C) \cap S(C') = \emptyset \). In order to prove it, one can reverse the above procedure and uniquely
Consequently, we can uncover all edges $g_1, g_2, \ldots, g_{2k-a-1}$ from $\hat{C}$. Note that we do not know the order in which the vertices of $e$ and $f$ are traversed by $C$.

To reconstruct $C$, we first find in $\hat{C}$ a unique $e \sim f$ path with no endpoint in $e \cap f$, say $Q$. From this we deduce that $u_1 = Q \cap e, v_k = Q \cap f$ and $w_k^{2k-a-1}$ is the last vertex on $Q$ before $v_k$. Now we start at $v_k$ and follow $\hat{C}$ in the direction opposite to $w_k^{2k-a-1}$. Before we reach $u_1$ we will intersect $f \cup e$ exactly $2k - a - 2$ times. This way we restore the vertices $v_{k-1}, v_{k-2}, \ldots, v_{a+1}, u_k, u_{k-1}, \ldots, u_2$ (in the order of appearance on $C$). Note that every one of these vertices is adjacent to two vertices $w_{k-1}^j$ and $w_k^j$ for some $1 \leq j \leq 2k - a - 2$. Consequently, we can uncover all edges $g_i$ and hence $C$ itself.

**Construction 2:** for $|e \cap f| \leq 1$.

Here we show a stronger result, namely, we construct a family $S(C)$ of size $\Omega(n^{2(k-1)})$. Let $e = (u_1, \ldots, u_k)$ and $f = (v_1, \ldots, v_k)$. Note that it might happen that $u_k = v_1$ if $|e \cap f| = 1$. Let $P$ be a segment of vertices between $e$ and $f$ of size $\Omega(n)$. For given two vertices $x$ and $y$ denote by $d(x, y)$ the number of vertices on $P$ in the segment between $x$ and $y$. Now we select $2(k-1)$ vertex disjoint edges $g_1, \ldots, g_{2(k-1)}$ from $P$ so that $C$ is of the form $e \sim f \sim g_1 \sim \cdots \sim g_{2(k-1)}$ and

$$d(v_k, w_{k-1}^1) < d(w_k^1, w_{k-1}^2) + 1 < d(w_k^2, w_{k-1}^3) + 1 < \cdots < d(w_k^{k-2}, w_{k-1}^{k-1}) + 1. \quad (6)$$

This way we fix an orientation of $C$. (The above sequence of inequalities will be needed later to establish the orientation of $C$ from $\hat{C}$.) Clearly, we have $\Omega(n^{2(k-1)})$ choices for $g_i$'s.

Let $g_i = (w_i^1, \ldots, w_i^j)$ for $1 \leq i \leq 2(k-1)$, where we list the vertices of $g_i$ in the order of appearance on $P$. In order to create a cycle $\hat{C} \in S_C$, we remove all edges which contain at least one vertex from $(e \cup f) \setminus \{u_1, v_k\}$ and also all edges whose first vertex (in the order induced by $C$) is $w_i^j$ for $1 \leq i \leq 2(k-1)$ and $j = 1, \ldots, k-1$. After this removal, the vertices in the set $\{u_2, \ldots, u_{k-1}, v_2, \ldots, v_{k-1}\}$ become isolated and the remains of the cycle $C$ form a collection of vertex disjoint paths $v_k \sim w_{k-1}^1, w_k^1 \sim w_{k-1}^2, w_k^2 \sim w_{k-1}^3, \ldots, w_k^{2(k-1)} \sim u_1$, and $u_k \sim v_1$. (The latter may be degenerated to the set of isolated vertices.)

To create $\hat{C}$, we connect the above paths by absorbing the isolated vertices. Formally, we define $\hat{C}$ as the following sequence of vertices (see Fig. 3):

$$v_1, u_{k-1}^1, w_{k-2}^1, \ldots, w_1^1 \sim v_k, u_k$$

$$\sim w_1^2, w_2^2, \ldots, w_{k-1}^2, v_{k-1}^2, w_k^2$$

$$\sim w_1^3, w_2^3, \ldots, w_{k-1}^3, v_{k-2}^3, w_k^3$$

$$\sim \cdots$$

$$\sim w_1^{k-1}, w_2^{k-1}, \ldots, w_{k-1}^{k-1}, v_2, w_k^{k-1}$$

$$\sim w_1^k, w_2^k, \ldots, w_{k-1}^k, u_{k-1}, w_k^k$$

$$\sim w_1^{k+1}, w_2^{k+1}, \ldots, w_{k-1}^{k+1}, u_k, w_k^{k+1}$$

$$\sim \cdots$$

$$\sim w_1^{2k-3}, w_2^{2k-3}, \ldots, w_{k-1}^{2k-3}, u_2, w_k^{2k-3}$$

$$\sim w_1^{2k-2}, w_2^{2k-2}, \ldots, w_{k-1}^{2k-2}, u_1 \sim w_k^{2k-2}, u_k \sim v_1.$$
It is easy to check that every new edge intersects a vertex from $e \cup f$. Thus, $\tilde{C} \setminus C$ contains no edge from any pair of edges belonging to $X$. Moreover, note that different choices of $g_i$ yield different cycles $\tilde{C}$. Thus, $|S(C)| = \Omega(n^{2(k-1)})$.

It remains to show that for any two tight Hamilton cycles $C \neq C' \in \mathcal{C}_1$ we have $S(C) \cap S(C') = \emptyset$. In order to prove it, one can reverse the above procedure and uniquely determine $C$ and the edges $g_1, g_2, \ldots, g_{2(k-1)}$ from $\tilde{C}$. Note that we do not know the order in which the vertices of $e$ and $f$ are traversed by $C$.

Note that there are exactly two $e \rightsquigarrow f$ paths in $\tilde{C}$, say $Q_1$ and $Q_2$. One with $u_k$ and $v_1$ as its endpoints and the second one with $u_{k-1}$ and $v_2$ as the endpoints. Our goal is to determine vertex $v_1$. Once this is known then as in Construction 1 we can uncover all edges $g_i$ and hence $C$ itself.

We assume for a while that $v_1 = Q_1 \cap f$. Then we start at $v_1$ and follow $\tilde{C}$ in the direction opposite to the second endpoint of $Q_1$. Before we reach edge $e$ we will intersect edge $f$ exactly $k-1$ times. This way we pretend that we restore vertices $v_k, v_{k-1}, \ldots, v_2$ (in the order of appearance). Let $\tilde{d}(x,y)$ be the number of vertices on $\tilde{C}$ between vertices $x$ and $y$. Note that $d(v_k, w_{k-1}^1) = \tilde{d}(v_1, v_k) - 1$ and $d(w_k^1, w_{k-1}^{j+1}) = \tilde{d}(v_{k-j+1}, v_{k-j}) - 2$ for $1 \leq j \leq k-2$. Now we check if (6) holds. If so then $Q_1$ is really the path with endpoints $u_k$ and $v_1$; otherwise $Q_2$ is the one.

References


