Upper bounds on the minimum size of Hamilton saturated hypergraphs

Andrzej Ruciński*  Andrzej Žak †
Department of Discrete Mathematics  Faculty of Applied Mathematics
Adam Mickiewicz University  AGH University of Science and Technology
Poznań, Poland  Kraków, Poland
rucinski@amu.edu.pl  zakandrz@agh.edu.pl

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Abstract
For $1 \leq \ell < k$, an $\ell$-overlapping $k$-cycle is a $k$-uniform hypergraph in which, for some cyclic vertex ordering, every edge consists of $k$ consecutive vertices and every two consecutive edges share exactly $\ell$ vertices.

A $k$-uniform hypergraph $H$ is $\ell$-Hamiltonian saturated if $H$ does not contain an $\ell$-overlapping Hamiltonian $k$-cycle but every hypergraph obtained from $H$ by adding one edge does contain such a cycle. Let $\text{sat}(n, k, \ell)$ be the smallest number of edges in an $\ell$-Hamiltonian saturated $k$-uniform hypergraph on $n$ vertices. In the case of graphs Clark and Entringer showed in 1983 that $\text{sat}(n, 2, 1) = \lceil \frac{3n}{2} \rceil$. The present authors proved that for $k \geq 3$ and $\ell = 1$, as well as for all $0.8k \leq \ell \leq k - 1$, $\text{sat}(n, k, \ell) = \Theta(n^\ell)$. In this paper we prove two upper bounds which cover the remaining range of $\ell$. The first, quite technical one, restricted to $\ell \geq \frac{k+1}{2}$, implies in particular that for $\ell = \frac{3}{2}k$ and $\ell = \frac{3}{4}k$ we have $\text{sat}(n, k, \ell) = O(n^{\ell+1})$. Our main result provides an upper bound $\text{sat}(n, k, \ell) = O(n^{(k+\ell)/2})$ valid for all $k$ and $\ell$. In the smallest open case we improve it further to $\text{sat}(n, 4, 2) = O(n^{14/5})$.

1 Introduction

A hypergraph $H$ is a pair $H = (V, E)$ where $V$ is a set of elements called vertices, and $E$ is a set of non-empty subsets of $V$ called edges. If every edge of $H$ has exactly $k$ vertices, then $H$ is called a $k$-uniform hypergraph or a $k$-graph. In what follows we will often identify $H$ with its set of edges.

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Given integers $1 \leq \ell < k$, we define an $\ell$-overlapping $k$-cycle as a $k$-graph in which, for some cyclic ordering of its vertices, every edge consists of $k$ consecutive vertices, and every two consecutive edges (in the natural ordering of the edges induced by the ordering of the vertices) share exactly $\ell$ vertices. The notion of an $\ell$-overlapping $k$-path is defined similarly, that is, with vertices ordered $v_1, \ldots, v_s$, the edges of the path are $\{v_1, \ldots, v_\ell\}, \{v_\ell, v_{\ell+1}, \ldots, v_\ell\}, \ldots, \{v_s, v_{s-\ell}, \ldots, v_s\}$. Note that the number of edges of an $\ell$-overlapping $k$-cycle with $s$ vertices is $s/(k-\ell)$ (and thus, $s$ is divisible by $k-\ell$). Similarly, it can be easily seen that the number of vertices $s$ of an $\ell$-overlapping $k$-path equals $\ell$ modulo $k-\ell$.

We denote an $\ell$-overlapping $k$-cycle on $s$ vertices by $C_s^{(k,\ell)}$. We further denote by $g := g(k, \ell)$ the number of vertices between any two consecutive disjoint edges belonging to an $\ell$-overlapping path (or cycle) and notice that

$$0 \leq g = \left\lfloor \frac{k}{k-\ell} \right\rfloor (k-\ell) = k - \ell < k - \ell < k,$$

and that $g = 0$ if and only if $k - \ell$ divides $k$.

An $\ell$-overlapping Hamiltonian $k$-cycle in a $n$-vertex $k$-graph $H$ is defined as any $\ell$-hypergraph of $H$ isomorphic to $C_n^{(k,\ell)}$. If $H$ contains an $\ell$-overlapping Hamiltonian $k$-cycle then $H$ itself is called $\ell$-Hamiltonian.

Given a $k$-graph $H$ and a $k$-element set $e \in H^c$, where $H^c = \binom{V}{k} \setminus H$ is the complement of $H$, we denote by $H + e$ the hypergraph obtained from $H$ by adding $e$ to its edge set. A $k$-graph $H$ is $\ell$-Hamiltonian saturated, $1 \leq \ell \leq k-1$, if $H$ is not $\ell$-Hamiltonian but for every $e \in H^c$ the $k$-graph $H + e$ is such. The largest number of edges in an $\ell$-Hamiltonian saturated $k$-graph on $n$ vertices is called the Turán number for the cycle $C_n^{(k,\ell)}$. In [2] this number has been determined in terms of the Turán number of a $(k-1)$-uniform path with a constant number of vertices.

In this paper we are interested in the other extreme. For $n$ divisible by $k - \ell$, let $\text{sat}(n,k,\ell)$ be the smallest number of edges in an $\ell$-Hamiltonian saturated $k$-graph on $n$ vertices. In the case of graphs, Clark and Entringer proved in 1983 that $\text{sat}(n,2,1) = \left\lceil \frac{3n}{2} \right\rceil$ for $n \geq 52$.

For $k$-graphs with $k \geq 3$ the problem was first mentioned in [3, 4]. It seems to be quite hard to obtain such precise results as for graphs. Therefore, the emphasis has been put on the order of magnitude of $\text{sat}(n,k,\ell)$. The present authors proved in [5] that for $k \geq 3$ and $\ell = 1$, as well as for all $0.8k \leq \ell \leq k - 1$,

$$\text{sat}(n,k,\ell) = \Theta(n^{\ell}),$$

see also [6] for the case $\ell = k - 1$. On the other hand, we have the easy lower bound ([5, Prop. 2.1])

$$\text{sat}(n,k,\ell) = \Omega \left( n^{\ell} \right).$$

The facts that (2) holds for very small and very large (with respect to $k$) values of $\ell$ and that no better lower bound is known suggest, as conjectured already in [5], that (2) holds for all $1 \leq \ell \leq k - 1$ and $k \geq 2$. 

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Conjecture 1. For all $k \geq 2$ and $1 \leq \ell \leq k - 1$,
\[ \text{sat}(n, k, \ell) = O\left(n^\ell\right). \]

Our first result provides an upper bound on $\text{sat}(n, k, \ell)$ higher than the conjectured $O(n^\ell)$, but for a broader range of $\ell$ than in [5].

Theorem 1. For all $k \geq 3$ and $\ell \geq \frac{k+1}{2}$
\[ \text{sat}(n, k, \ell) = O\left(n^{\ell+2g+1}\right). \]

Of course, this bound is good only when $g$ is small, and when $g = 0$ it is only by a factor of $n$ worse than the conjectured optimum. All cases of Theorem 1 which are not covered by the result from [5], but for which $g = 0$, are given in the following corollary.

Corollary 2. For every $k$ divisible by three and $\ell = \frac{2}{3}k$, as well as for every $k$ divisible by four and $\ell = \frac{3}{4}k$, we have $\text{sat}(n, k, \ell) = O(n^{\ell+1})$.

In the remaining range of $\ell$, that is, for $2 \leq \ell \leq k/2$, nothing else than the trivial upper bound
\[ \text{sat}(n, k, \ell) = O(n^k) \]
have been known. Our main result in this paper provides a first, non-trivial, general upper bound on $\text{sat}(n, k, \ell)$.

Theorem 3. For all $k \geq 3$ and $2 \leq \ell \leq k - 1$,
\[ \text{sat}(n, k, \ell) = O\left(n^{(k\ell)/2}\right). \]

One consequence of Theorem 3, combined with the case $\ell = k - 1$ of (2), is that for all $\ell$ and $k$ we have
\[ \text{sat}(n, k, \ell) = O\left(n^{k-1}\right). \]
In view of Theorem 3, the bound in Theorem 1 is not overwritten only when $\ell + 2g + 1 \leq \frac{k+\ell-1}{2}$, equivalently, when $g \leq (k - \ell - 1)/4$. Theorems 1 and 3 are proved, respectively, in Sections 3 and 4. In the smallest open case, $k = 4$, $\ell = 2$, we improve Theorem 3 a bit by showing the following result in Section 5.

Theorem 4. $\text{sat}(n, 4, 2) = O\left(n^{14/5}\right)$.

Our proofs expand and refine a general approach to this type of problems first developed in [6] and modified in [5]. In short, we begin with constructing two $k$-graphs, $H'$ and $H''$, such that $H'$ is not $\ell$-Hamiltonian, while $H'' \supset H'$ contains some “trouble-making” edges. Then we define $H$ as a maximal non-$\ell$-Hamiltonian $k$-graph satisfying $H' \subseteq H \subseteq H''$. It then remains to show that for every $e \not\in H$, $H + e$ is $\ell$-Hamiltonian, but, what is crucial, in doing so we may restrict ourselves to $e \not\in H''$.

In [6] the constructions of $H'$ and $H''$ were based on a special partition of the vertex set, while in [5] we used blow-ups of sparse Hamiltonian saturated graphs. In this paper we return to both these ideas: we use the approach from [5] in the proof of Theorem 1, and the approach from [6] in the proofs of Theorems 3 and 4.
2 Preliminaries

Our proofs utilize the following special construction of a $k$-graph. Given a partition of the vertex set $V = \bigcup_{i=1}^{h} U_i$, for a subset $S \subseteq V$, let

$$tr(S) = \{ i : U_i \cap S \neq \emptyset \}$$

and

$$\min(S) = \min\{ i : i \in tr(S) \} = \min\{ i : U_i \cap S \neq \emptyset \}.$$

Let

$$H_{k,\ell}(U_1, \ldots, U_h) := H_{k,\ell} = \left\{ e \in \binom{V}{k} : \left| e \cap U_{\min(e)} \right| \geq k - \ell + 1 \right\}.$$

For further use, note that

$$|tr(e)| \leq \ell \quad \text{for every } e \in H_{k,\ell}. \quad (3)$$

For $i = 1, \ldots, h$, let

$$C_i = \{ e \in H_{k,\ell} : \min(e) = i \}.$$

Obviously, $H_{k,\ell} = C_1 \cup \cdots \cup C_h$.

Define an $\ell$-component of a $k$-graph $H$ as a minimal subset of edges $C \subseteq H$ such that for all $e \in C$ and $f \in H \setminus C$, we have $|e \cap f| < \ell$.  

**Proposition 5.** For each $i = 1, \ldots, h$, the set $C_i$ is an $\ell$-component of $H_{k,\ell}$.

**Proof.** By the definition of $H_{k,\ell}$, for every $e \in C_i$ and $f \in C_j$, where $i < j$, we have $|e \cap U_i| \geq k - \ell + 1$ and $f \cap U_i = \emptyset$, and so $|e \cap f| < \ell$. Moreover, for every $e \in C_i$ there is an $f \in C_i$, $f \neq e$ such that $|e \cap f| \geq k - 1 \geq \ell$ (just switch one vertex without violating the membership in $C_i$), so that $C_i$ satisfies the minimality condition in the definition of an $\ell$-component. \hfill $\Box$

Since every $\ell$-overlapping $k$-path in a $k$-graph $H$ must be entirely contained in one the $\ell$-components of $H$, we have the following corollary of Proposition 5.

**Corollary 6.** For every $\ell$-overlapping $k$-path $P$ in $H_{k,\ell}$ there is an $i \in \{1, \ldots, h\}$ such that $P \subseteq C_i$, or equivalently, for every edge $e$ of $P$, we have $\min(e) = i$.

We now investigate the maximum length of an $\ell$-overlapping $k$-path in $C_i$, $i < h$, which traverses through exactly $x$ vertices of $U_i$. Our next, purely combinatorial, result provides an easy upper bound, independent of $\ell$. Given a positive integer $x$, let $A$ and $B$ be two disjoint sets, with $|A| = x$ and $|B| = \infty$. Let $\nu(x) = \max_P \left| V(P) \right|$, where the maximum is taken over all $\ell$-overlapping paths $P$ with $A \subseteq V(P) \subseteq A \cup B$ and $|e \cap A| \geq k - \ell + 1$ for all $e \in P$.

**Proposition 7.** For every $x \geq k - 2$, we have $\nu(x) \leq kx$. 

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Proof. Suppose there is a path $P$ with $A \subset V(P) \subset A \cup B$, $|e \cap A| \geq k - \ell + 1$ for all $e \in P$, and $|V(P)| \geq kx + 1$. Let us view $V(P)$ as a binary sequence, where each vertex of $A$ is replaced by symbol $a$ and each vertex of $V(P) \cap B$ is replaced by symbol $b$. If there is a pair of consecutive symbols $a$ in the sequence then, by averaging, there is a run (=a sequence of consecutive symbols) of at least

$$\frac{(k-1)x+1}{x} > k-1,$$

that is, of at least $k$ symbols $b$. But then there is an edge of $P$ with at most $k - \ell$ vertices of $A$ – a contradiction. If, on the other hand, there are no consecutive symbols $a$ in the sequence then, again by averaging, there is a run of at least

$$\frac{(k-1)x+1}{x+1} > k-2,$$

that is, of at least $k - 1$ symbols $b$ (here we use the assumption $x \geq k-2$). Thus, there is a segment $b \cdots bab$ where the run of $b$'s is of length $k-1$. The first (from the left) edge of $P$ whose leftmost end is in this run may have at most $k - \ell$ symbols $a$ – a contradiction, again.

We also have the following lower bound on $\nu(x)$.

**Proposition 8.** For every $x \geq (k-3)(k-1)$

$$\nu(x) \geq x + \left\lfloor \frac{x}{k-1} \right\rfloor + 3 - k.$$

**Proof.** Let a sequence $Q$ begin with a vertex in $B$ and then traverse, alternately, groups of $k - 1$ vertices of $A$ followed by one vertex of $B$ until fewer than $k - 1$ vertices of $A$ are left. The remaining vertices of $A$ are placed all at one end of $Q$. Clearly, every $k$-tuple of consecutive vertices of $Q$ contains $k - 1 \geq k - \ell + 1$ vertices of $A$. To turn $Q$ into an $\ell$-overlapping path, the number of vertices of $Q$ must equal $\ell$ modulo $k - \ell$. Therefore, we may be forced to drop up to $k - \ell - 1 \leq k - 2$ vertices of $B$ from $Q$. This is possible as

$$|Q \cap B| = \left\lfloor \frac{x}{k-1} \right\rfloor + 1 \geq k - 2,$$

by our assumption on $x$. The obtained path has the required properties and the claimed number of vertices.

Note that $\nu(x)$ is a nondecreasing function of $x$ (just replace any vertex of $B$ with a new vertex of $A$). Our next observation shows that it cannot increase too fast.

**Proposition 9.** For all $x \geq 1$ we have $\nu(x-1) \geq \nu(x) - k$. 
Proof. \textup{Consider a longest path } P \textup{ of length } \nu(x) \textup{ and remove its first (from the left) } s \textup{ vertices, where } \ell \leq s \leq k \textup{ and } s \equiv \nu(x) \pmod {k - \ell}. \textup{ As there must be a vertex of } A \textup{ among the first } \ell \textup{ vertices of any edge, the remaining path } P' \textup{ satisfies } x' := |V(P') \cap A| \leq x - 1 \textup{ and, by the monotonicity of } \nu(x) \textup{ we have}
$$
\nu(x) - k \leq \nu(x) - s \leq \nu(x') \leq \nu(x - 1).
$$
\qed

Returning to the hypergraph \( H_{k,\ell} \), Propositions 7-9 imply the following corollary.

\textbf{Corollary 10.} Let \( i < h \), \( k^2 \leq x \leq |U_i| \), \( A \subseteq U_i \), \( |A| = x \), and \( B \subseteq \bigcup_{j \geq i} U_j \), \( |B| \geq (k - 1)x \). Then the length of a longest path \( P \) in \( C_i \) such that \( A \subseteq V(P) \subset A \cup B \) equals \( \nu(x) \). Moreover, we have \( \nu(x) - k \leq \nu(x - 1) \leq \nu(x) \) and
$$
\frac k{k-1}x - k < \nu(x) \leq kx.
$$

In addition to the basic construction \( H_{k,\ell} \), the proof of Theorem 1 relies on the notion of a (hypergraph) blow-up of a graph which will be defined soon. First, however, we recall a simple fact about graphs proved in \([5, \text{Fact 2.2}]\). For a graph \( G \), let \( c(G) \) denote the number of components of \( G \). Given a subset \( T \subseteq V(G) \), let \( G[T] \) be the subgraph of \( G \) induced by \( T \).

\textbf{Fact 11} ([5]). \textit{Let } \( k, \ell \), \textit{and } \( \Delta \) \textit{ be constants, and for } \( h = 1, 2, \ldots, \) \textit{let } \( G_h \) \textit{ be a graph with } \( h \) \textit{ vertices and } \( \Delta(G_h) \leq \Delta \). \textit{Then the number of } \( k \)-\textit{element subsets } \( T \subseteq V(G_h) \) \textit{ with } \( c(G[T_h]) \leq \ell \) \textit{ is } \( O(h^\ell) \).

Given a graph \( G \) and an integer sequence \( \mathbf{a} = (a_1, \ldots, a_h) \), the \( \textbf{a}\text{-blow-up} \) of \( G \) is the \( k\)-graph \( H := H[G] \) with
$$
V(H) = \bigcup_{i=1}^h U_i, \quad |U_i| = a_i,
H = \bigcup_{ij \in G} K^{(k)}(U_i \cup U_j)
$$
where \( K^{(k)}(U) \) is the complete \( k \)-graph on \( U \) and the sets \( U_i \) are pairwise disjoint. For a subset \( S \subseteq V(H) \), let \( tr(S) = \{ i \in V(G) : U_i \cap S \neq \emptyset \} \).

Furthermore, set
$$
c(S) = c(G[tr(S)]).
$$
The following immediate corollary of Fact 11 has been already noted in \([5, \text{Cor. 2.3}]\).

\textbf{Corollary 12} ([5]). \textit{Let } \( a_1, \ldots, a_h, k, \ell, \) \textit{and } \( \Delta \) \textit{ be constants. If } \( \Delta(G_h) \leq \Delta \) \textit{ and } \( H_h = H[G_h] \) \textit{ is the } \( \textbf{a}\text{-blow-up} \) \textit{ of } \( G_h \) \textit{ then the number of } \( k \)-\textit{element subsets } \( S \subseteq V(H_h) \) \textit{ with } \( c(S) \leq \ell \) \textit{ is } \( O(h^\ell) \).

In order to facilitate the reading of the paper, the most frequent notation has been summarized in Table 1.

\textbf{Table 1.}
\[ g(k, \ell) = \left\lceil \frac{k}{k-\ell} \right\rceil (k - \ell) - k \]

- **H**: a \( k \)-graph
- **G**: an auxiliary graph
- **V(H)**: \( \bigcup_{i=1}^{h} U_i \)
- **V(G)**: \{1, \ldots, h\}
- **n**: \( |V(H)| \)
- **tr(S)**: \( \{i : U_i \cap S \neq \emptyset\} \)
- **min(S)**: \( \min\{i : S \cap U_i \neq \emptyset\} \)
- **min(S)**: \( \min\{i : (S \setminus U_{\min(S)}) \cap U_i \neq \emptyset\} \)
- **c(G)**: the number of components of \( G \)
- **c(S)**: \( c(G[tr(S)]) \)
- **\( H_{k,\ell} \)**: \( \{e \in \binom{V}{2} : |e \cap U_{\min(e)}| \geq k - \ell + 1\} \)
- **\( C_i \)**: \( \{e \in \mathbb{H}_{k,\ell} : \min(e) = i\} \)
- **\( \nu(x) \)**: \( \max\{|V(P)| : P \text{ is an } \ell\text{-overlapping path with } |V(P) \cap A| = x \text{ and } |e \cap A| \geq k - \ell + 1 \text{ for all } e \in P\} \)

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### 3 Proof of Theorem 1

In this section we prove Theorem 1, where the construction of an \( \ell \)-Hamiltonian saturated \( k \)-graph is based on a blow-up of a suitably chosen Hamiltonian saturated graph.

Our proof is a substantial modification of the proof of Theorem 1.1 in [5]. Specifically, we have made the range of \( \ell \) in (7) broader (it used to be \( 2k - \ell + 1 \leq a_i \leq 4\ell - 2k + 1 \)) and, at the same time, we altered the definition of \( \mathbb{H} \) (by introducing the cores \( U_i \)). In what follows, we assume that

\[ g \leq \frac{k - \ell - 1}{4}, \]  

since otherwise \( \ell + 2g + 1 \geq (k + \ell)/2 \) and Theorem 1 follows from Theorem 3.

We begin with a technical inequality.

**Proposition 13.** If \( \frac{k + 1}{2} \leq \ell \leq k - 1 \) then \( 2k - \ell - 2g - 2 \leq 2\ell - 2 \).

**Proof.** The inequality in question is equivalent to

\[ 3\ell + 2g \geq 2k, \]  

To prove (5), note that, by the assumptions on \( \ell \), there exists some integer \( a \geq 1 \) such that

\[ \frac{ak + 1}{a + 1} \leq \ell < \frac{(a + 1)k + 1}{(a + 1) + 1} \leq \frac{2ak + 1}{2a + 1}. \]
Then, by the lower bound on $\ell$,
\[ g = \left\lceil \frac{k}{k - \ell} \right\rceil (k - \ell) - k \geq \left\lceil \frac{k}{k - (ak + 1)/(a + 1)} \right\rceil (k - \ell) - k = \left\lceil \frac{k}{k - 1} \right\rceil (a + 1) (k - \ell) - k \geq (a + 2)(k - \ell) - k. \]
Hence, by the upper bound on $\ell$, we finally have
\[ 3\ell + 2g \geq (2a + 2)k - (2a + 1)\ell > 2k - 1, \]
which implies (5).

It follows from Proposition 13, as in [5], that every sufficiently large integer $n$ can be expressed as a sum
\[ n = a_1 + \cdots + a_h, \tag{6} \]
for some $h$, where
\[ 2k - \ell - 2 - 2g \leq a_i \leq 2\ell - 1, \quad i = 1, \ldots, h. \tag{7} \]
(This is because the range of $a_i$ in (7) has at least two consecutive values.)

Fix a large integer $n$ which is divisible by $(k - \ell)$ and let $a = (a_1, \ldots, a_h)$, where the $a_i$’s and $h$ are as in (7). Note that $n = \Theta(h)$. Let $G_h$ be an $h$-vertex Hamiltonian saturated graph with $\Delta(G_h) = O(1)$, and let
\[ H_1 = H[G_h] \]
be the $a$-blow-up $k$-graph of $G_h$ (see the definition in Section 2) with
\[ V = V(H_1) = \bigcup_{i=1}^h U_i, \quad \text{where} \quad |U_i| = a_i, \quad i = 1, \ldots, h. \]
Thus, by (6),
\[ |V| = n = \sum_{i=1}^h a_i. \]
It is easy to check that (4) implies that $a_i \geq k - \ell$, for all $i = 1, \ldots, h$. Fix a $(k - \ell)$-subset $U_i$ of $U_i$, $i = 1, \ldots, h$, and let
\[ H_2 = \left\{ e \in \left( {\binom{V}{k}} \right) : |e \cap U_{\min(e)}| \geq k - l + 1, e \supset U_{\min(e)} \text{ and } c(e) \geq g + 2 \right\}. \]
Since $H_2 \subseteq H_{k,\ell}$, by (3), for every $e \in H_2$ we have, in fact,
\[ 2 \leq g + 2 \leq c(e) \leq |\text{tr}(e)| \leq \ell. \tag{8} \]
(Note that (4) implies that, indeed, $g \leq \ell - 2$, which guarantees that $H_2$ is nonempty.) We have the following immediate consequence of the definition of $H_2$ and Corollary 6.
Corollary 14. If \( P \) is a path in \( H_2 \), then there is \( i \in \{1, \ldots, h\} \) such that for every \( e \in P \) we have \( |e \cap U_i| \geq k - \ell + 1 \) and \( e \supseteq U_i \). In particular, each path in \( H_2 \) has at most \( \left\lfloor \frac{k}{\ell} \right\rfloor \) edges.

Observe also that for each \( e \in H_1 \), the set \( tr(e) \) is either a vertex or an edge of \( G \). Consequently, \( c(e) = 1 \) and the \( k \)-graphs \( H_1 \) and \( H_2 \) are edge-disjoint. Set \( H' = H_1 \cup H_2 \)

**Lemma 15.** \( H' \) is not \( \ell \)-Hamiltonian.

**Proof.** Suppose that \( H' \) contains an \( \ell \)-Hamiltonian \( k \)-cycle \( C_H = (e_1, \ldots, e_m) \). Unlike in [5], the proof breaks only into two cases:

**Case 1.** \( C_H \subseteq H_1 \): We omit the proof in this case, as it is identical to Case 1 of the proof of Lemma 4.1 in [5] (Indeed that proof relied only on the assumption that \( a_i \leq 2 \ell - 1 \)).

**Case 2.** \( H_2 \cap C_H \neq \emptyset \): Let (w.l.o.g.) \( e_1, \ldots, e_{s-1} \) be a maximal segment in \( C_H \) of consecutive edges from \( H_2 \). By Corollary 14, \( s - 1 \leq \left\lfloor \frac{k}{\ell} \right\rfloor \) and there exists an index \( i \in \{1, \ldots, h\} \) such that

\[
e_1 \cap e_{s-1} \supseteq U_i, \quad \text{and thus} \quad |e_1 \cap e_{s-1}| \geq |U_i| = k - \ell.
\]

Let \( Z \) be the set of vertices that lie between \( e_m \) and \( e_s \) on \( C_H \). Formally,

\[
Z = \left( \bigcup_{t=1}^{s-1} e_t \right) \setminus (e_m \cup e_s).
\]

Then \( e_1 \subseteq e_m \cup Z \cup e_s \) and, consequently,

\[
\{i\} \subseteq tr(e_1) \subseteq tr(e_m) \cup tr(Z) \cup tr(e_s).
\]

What is more, \( e_m \cap U_i \neq \emptyset \) and \( e_s \cap U_i \neq \emptyset \). Since \( e_m \in H_1 \) and \( e_s \in H_1 \), by the definition of \( H_1 \), each of \( tr(e_m) \) and \( tr(e_s) \) is either the singleton \( \{i\} \) or an edge of \( G \) containing vertex \( i \). Hence, by (10), \( c(e_1) \leq 1 + |Z| \), which combined with the bound \( g + 2 \leq c(e_1) \) from the definition of \( H_2 \), yields

\[
|Z| \geq g + 1.
\]

This further implies that \( e_m \) and \( e_s \) are disjoint, but more importantly, that \( e_1 \) and \( e_s \) are disjoint too (since \( e_m \) and \( e_s \) cannot be consecutive disjoint edges). Thus, \( s \geq 3 \) and

\[
|Z| \leq 2(k - \ell) - |e_1 \cap e_{s-1}| \leq k - \ell,
\]

by (9). Note, however, that due to the structure of \( \ell \)-overlapping \( k \)-paths,

\[
|Z| = g + t(k - \ell) \quad \text{for some} \quad t \geq 0.
\]

Therefore, by (13), (12) and (11), \( |Z| = k - \ell \) (and \( g = 0 \)). Consequently, by (12), \( |e_1 \cap e_{s-1}| = k - \ell \), implying that, in fact, \( e_1 \cap e_{s-1} = Z = U_i \). But then (10) becomes

\[
\{i\} \subseteq tr(e_1) \subseteq tr(e_m) \cup tr(e_s),
\]
and hence, \( c(e_1) = 1 \) – a contradiction with the definition of \( H_2 \).

Let

\[
H'' = \left\{ e \in \binom{V}{k} : c(e) \leq \ell + 2g + 1 \right\}.
\]

Recall that \( H_1 = H[G_h] \) is the a-blow-up \( k \)-graph of a Hamiltonian saturated \( h \)-vertex graph \( G_h \). It means that for all \( e \in H_1 \) we have \( c(e) = 1 \), while, by (8), for all \( e \in H_2 \) we have \( c(e) \leq |tr(e)| \leq \ell \). Thus, \( H' = H_1 \cup H_2 \subseteq H'' \).

Finally, let \( H \) be a maximal non-\( \ell \)-Hamiltonian \( k \)-graph on \( V \) such that \( H' \subseteq H \subseteq H'' \).

In view of Lemma 15, \( H \) does exist. By Corollary 12,

\[
|H| \leq |H''| = O(n^{\ell+2g+1}). \tag{14}
\]

Thus, to complete the proof of Theorem 1, it remains to show the following lemma.

**Lemma 16.** For every \( e \in H', H + e \) is \( \ell \)-Hamiltonian.

**Proof.** By the maximality of \( H, H + e \) is \( \ell \)-Hamiltonian for each \( e \in H'' \setminus H \). Hence, we may restrict ourselves only to \( e \not\in (H'')^c \), that is, such that \( c(e) \geq \ell + 2g + 2 \). Let us fix one such \( e \). Let \( j_1, j_2, \ldots, j_{\ell+2g}, y \), and \( x = \min(e) \) belong to \( \ell + 2g + 2 \) different components of \( G[tr(e)] \) and satisfy

\[
\min\{j_1, j_2, \ldots, j_{\ell+2g}\} > y > x. \tag{15}
\]

Let \( r_x = |e \cap U_x| \) and \( r_y = |e \cap U_y| \). Note that, since \( |tr(e)| \geq c(e) \geq \ell + 2g + 2 \),

\[
\max\{r_x, r_y\} \leq \max_{1 \leq i \leq n} |e \cap U_i| \leq k - (|tr(e)| - 1) \leq k - \ell - 2g - 1. \tag{16}
\]

We will build an \( \ell \)-overlapping Hamiltonian cycle \( C_H \) in \( H + e \) using the Hamiltonian saturation of \( G_h \). Let \( (u_1, \ldots, u_n) \) be the vertices of \( V \) in the order as they will appear on the \( C_H \) under construction. Our goal is to define this ordering so that each segment of \( k \) consecutive vertices which begins at \( u_i \), where \( i \equiv 1 \mod k - \ell \), is an edge of \( H + e \). We will denote by \( e_1 \) the edge beginning at \( u_1 \), by \( e_2 \) – the edge beginning at \( u_1 + k - \ell \) and so on, until the last edge \( e_m \) of \( C_H \) which begins at \( u_{n-k+\ell+1} \), where \( m = \frac{n}{k-\ell} \).

To achieve our goal, we will first construct an \( \ell \)-overlapping path \( P \subseteq H_2 + e \), extending \( e \) in both directions, and using only the vertices of \( U_x \) and \( U_y \), one type at each end of \( e \). Then, we will connect the endsets of \( P \) by an \( \ell \)-overlapping path \( P' \subseteq H_1 \), covering all the remaining vertices and, thus, creating, together with \( P \), an \( \ell \)-overlapping Hamiltonian cycle in \( H + e \). The construction of \( P' \) will be facilitated by tracing a Hamiltonian path in \( G \) connecting \( x \) and \( y \).

To construct \( P \), let \( e_1 := e \) and order the vertices of \( e_1 = (u_1, \ldots, u_k) \) so that the first \( r_x \) vertices belong to \( U_x \), the last \( r_y \) vertices belong to \( U_y \), and the \( \ell - r_y \) vertices immediately preceding the \( r_y \) vertices of \( U_y \cap e_1 \) all belong to sets \( U_j \) with \( j > y \). (We know from (15) that there are more than enough such vertices in \( e_1 \).) In other words, we
request that
\[
\{u_1, \ldots, u_{r_x}\} \subset U_x, \tag{17}
\]
\[
\{u_{k-r_y+1}, \ldots, u_k\} \subset U_y, \tag{18}
\]
\[
\min \left( \{u_{k-\ell+1}, \ldots, u_{k-r_y}\} \right) > y. \tag{19}
\]

The remaining vertices of \(e_1\) are labeled arbitrarily by \(u_{r_x+1}, \ldots, u_{k-\ell}\).

Our plan is to extend \(e_1\) in either direction, but only for as long as the new edges still intersect \(e_1\). This means that we will have in \(P\) precisely
\[
\kappa := \left\lfloor \frac{l}{k-\ell} \right\rfloor
\]
new edges, and thus, precisely
\[
\kappa(k - \ell) = g + \ell
\]
new vertices on each side of \(e_1\), where the last equality follows from (1).

Formally, we set
\[
V(P) = \{u_{n-\ell-g+1}, \ldots, u_n, u_1, \ldots, u_k, u_{k+1}, \ldots, u_{k+g+\ell}\}
\]
and
\[
E(P) = \{e_1\} \cup \{e_{m+1-i} : i = 1, \ldots, \kappa\} \cup \{e_{1+i} : i = 1, \ldots, \kappa\},
\]
where, recall, the edge \(e_j\) begins at the vertex \(u_{1+(j-1)(k-\ell)}\).

We request that all vertices of \(P\) to the left of \(e_1\) belong to \(U_x\) and all vertices to the right of \(e_1\) belong to \(U_y\), that is,
\[
\{u_{n-\ell-g+1}, \ldots, u_n, u_1, \ldots, u_{r_x}\} \subseteq U_x \quad \text{and} \quad \{u_{k-r_y+1}, \ldots, u_k, u_{k+1}, \ldots, u_{k+g+\ell}\} \subseteq U_y. \tag{20}
\]

This is possible, since, by (16) and (7).
\[
\min (|U_x \setminus e|, |U_y \setminus e|) \geq 2k - \ell - g - 2 - (k - \ell - 2g - 1) = k + g - 1 \geq \ell + g.
\]

We also request that
\[
\{u_{n-k+\ell+1}, \ldots, u_{r_x}\} \supseteq \overline{U_x} \quad \text{and} \quad \{u_{k-r_y+1}, \ldots, u_{2k-\ell}\} \supseteq \overline{U_y}. \tag{21}
\]

This can be easily accommodated, as each of these sets contains precisely \(k - \ell\) vertices from outside of \(e_1\). Note that \(P\) is, trivially, an \(\ell\)-overlapping path in the complete \(k\)-graph on \(V\). We will show that, in fact, \(P \subseteq H_2 + e\).

Suppose first that \(m+1 - \kappa \leq j \leq m\). Then, by the definition of \(x\), \(\min(e_j) = x\). By our construction (see (17), (20), and (21)), \(|e_{j} \cap U_x| \geq k - \ell + 1\) and \(e_{j} \supseteq \overline{U_x}\). The same is true for \(e_j\) with \(j = 2, \ldots, \kappa + 1\), if we replace \(x\) by \(y\) (see (18), (19), (20), and (21)).

To conclude that \(P \subseteq H_2 + e\), it remains to show that \(c(e_j) \geq g + 2\) for each \(e_j, j \neq 1\). As, clearly, \(|e_j \setminus e_1| \leq \ell + g\), we also have
\[
|e_1 \setminus e_j| \leq \ell + g. \tag{22}
\]
Trivially, $c(e_1) \leq c(e_1 \setminus e_j) + c(e_1 \cap e_j)$. Moreover, $tr(e_j) = tr(e_1 \cap e_j)$. Therefore, by the choice of $e = e_1$ and (22),

$$c(e_j) = c(e_1 \cap e_j) \geq c(e_1) - c(e_1 \setminus e_j) \geq c(e_1) - |e_1 \setminus e_j| \geq \ell + 2g + 2 - (\ell + g) = g + 2.$$ 

Thus $e_j \in H_2$ for each $e_j \in P$, $j \neq 1$.

Now we will build the rest of $C_H$ using only the edges of $H_1$. Recall that $x$ and $y$ belong to different components of $tr(e)$ and, hence, $xy \not\in G$. Therefore, by the Hamiltonian saturation of $G$, there is a Hamiltonian path $Q = (v_1 = y, v_2, \ldots, v_{h-1}, v_h = x)$ from $y$ to $x$ in $G$. We connect the two $\ell$-element endsets of $P$ by an $\ell$-overlapping path $P' = (e_{m-g}, \ldots, e_{m-2})$ in $H_1 \subseteq H$ which, by tracing $Q$, “swallows” all the remaining $n - |V(P)|$ vertices of $V$.

Set $U'_v = U_v \setminus V(P)$, $v \in V(G)$, and 

$$R := \bigcup_{v \in V(G)} U'_v.$$ 

Observe that 

$$|R| = n - |V(P)| = n - 2\kappa(k - \ell) - k = n - 2(g + \ell) - k.$$ 

Let us order the elements $R$ so that all elements of $U'_{v_i}$ precede all elements of $U'_{v_i+1}$, for $i = 1, \ldots, h - 1$, and denote this ordering by $(u_{k+g+i+1}, \ldots, u_{n-g-\ell})$. The vertex set of $P'$ is then defined as 

$$V(P') = \{u_{k+g+1}, \ldots, u_{k+g+\ell}, u_{k+g+\ell+1}, \ldots, u_{n-g-\ell}, u_{n-g-\ell+1}, \ldots, u_n\}.$$ 

Note that for $v \not\in \{x, y\}$, by (7) and (16), 

$$|U'_v| \geq |U_v| - (k - \ell - 2g - 1) \geq k - 1.$$ 

Hence, every edge of $P'$ stretches over at most two sets $U_v$ and each such two sets are always indexed by adjacent vertices of $G$. This implies that $P' \subseteq H_1$. 

\[ \square \]

4 Proof of Theorem 3

In this section we prove Theorem 3, where the construction of an $\ell$-Hamiltonian saturated $k$-graph is based on a special partition of the vertex set into $q + 1$ sets $U_1, \ldots, U_{q+1}$ ($q$ to be chosen), and the associated with it notion of the hypergraph $H_{k,\ell}(U_1, \ldots, U_{q+1})$, introduced at the beginning of Section 2.

Recall that the function $\nu(x)$ has been defined in Section 2. Given a large integer $n$ divisible by $k - \ell$, choose integers $\alpha = \Theta\left(n^{1/2}\right)$, $\beta = \Theta\left(n^{1/2}\right)$, $p = \Theta\left(n^{1/2}\right)$, and 

$$q = \left\lceil \frac{p(k + 2g) + (p - 1)\nu}{\alpha} \right\rceil + 2,$$ 

(23)
where \( g = g(k, \ell) \) is given by (1) and \( \nu := \nu(\alpha) \), such that
\[
\alpha \geq 10k^3p, \tag{24}
\]
\[
\beta \geq k\alpha,
\]
and
\[
n = (q - 1)\alpha + \beta + p(k - 2) + k - 3. \tag{25}
\]

To see that such a choice is feasible, one may set, for instance, \( \alpha = \lceil 2k^2\sqrt{n} \rceil \). Recall that, by Proposition 7, \( \alpha \leq \nu \leq k\alpha \). Next, choose \( p = \lfloor n/\nu \rfloor - k - 1 \). Then, first of all, (24) holds. Furthermore, using (23) and the estimates \( g \leq k \), \( 2p \geq k - 3 \), and \( 4kp \leq \alpha \) among others, we can sandwich the quantity
\[
n - \beta = (q - 1)\alpha + p(k - 2) + k - 3
\]
as follows:
\[
n - (k + 3)\nu \leq \nu(p - 1) \leq n - \beta \leq 4kp + \alpha + n - (k + 2)\nu \leq n - k\alpha.
\]
Thus, there exists an integer \( \beta, k\alpha \leq \beta \leq (k + 3)\alpha \), which satisfies (25). Note that, in particular, by (23) and Proposition 8,
\[
q \geq p + 2k + 1. \tag{26}
\]
Let
\[
V = \bigcup_{i=1}^{q+1} U_i,
\]
where
\[
|U_i| = \alpha \quad \text{for} \quad i = 1, \ldots, q - 1, \quad |U_q| = \beta \quad \text{and} \quad |U_{q+1}| = p(k - 2) + k - 3,
\]
and all sets \( U_i, i = 1, \ldots, q + 1 \), are pairwise disjoint.

We begin our construction of the required \( \ell \)-Hamiltonian saturated \( k \)-graph \( H \), by letting
\[
H_1 = H_{k,\ell}(U_1, \ldots, U_{q+1}).
\]
Recall from Section 2 that \( H_1 \) breaks naturally into \( q + 1 \) \( \ell \)-components, that is, \( H_1 = C_1 \cup \cdots \cup C_{q+1} \). Thus, every path in \( H_1 \) is entirely contained in some \( C_i \), and, by Corollary 10, for all \( i \leq q - 1 \) such paths are no longer than \( k\nu \leq k^2\alpha \). On the other hand, by the definition of \( C_i \), the vertex set of every path contained in \( C_q \cup C_{q+1} \) must be a subset of \( U_q \cup U_{q+1} \). Therefore, in view of our assumptions on \( \beta, p \) and \( \alpha \), we have the following conclusion.

**Corollary 17.** The length of a longest path in \( H_1 \) is \( O(\sqrt{n}) \). In particular, \( H_1 \) is not \( \ell \)-Hamiltonian. \( \square \)
Following the outline described in the Introduction, we build a $k$-graph $H'$ by slightly enriching $H_1$, but so that it still remains non-$\ell$-Hamiltonian. Let

$$H_2 = \left\{ e \in \binom{V}{k} : |e \cap U_{q+1}| \geq k - 2 \right\}$$

and $H' = H_1 \cup H_2$.

**Lemma 18.** $H'$ is not $\ell$-Hamiltonian.

**Proof.** Suppose that $C$ is an $\ell$-overlapping Hamiltonian cycle in $H'$. Let $M$ be a maximal set of disjoint edges in $C \cap H_2$. By Corollary 17, $M \neq \emptyset$. Set $t := |M|$. Since

$$|U_{q+1}| = p(k - 2) + k - 3 \leq (p + 1)(k - 2),$$

we have $t \leq p$.

From $C$ we now extract $t$ vertex disjoint paths, all contained in $H_1$, as follows. For every $e \in M$, denote by $N(e)$ the union of the set of vertices of $e$, the set of $g$ consecutive vertices lying just before $e$, and the set of $g$ consecutive vertices lying just after $e$ (here, ‘before’ and ‘after’ refer to an arbitrarily fixed direction of traversing $C$). Let $W = \bigcup_{e \in M} N(e)$. Then $C[V \setminus W]$ consists of at most $t$ paths (we treat a nonempty set of fewer than $k$ consecutive isolated vertices as a single trivial path). Observe that

$$|W| \leq t(k + 2g).$$

Since each obtained path $P$ is contained in $H_1$, either $\min(V(P)) \leq q - 1$ or $V(P) \subseteq U_q \cup U_{q+1}$. If all $t$ paths are of the former kind, then their total number of vertices is at most $t\nu$, and otherwise, it is at most $(t - 1)\nu + |U_q| + |U_{q+1}|$. Note that, since $|U_q| = \beta \geq k\alpha \geq \nu$, we have

$$\max\{t\nu, (t - 1)\nu + |U_q| + |U_{q+1}|\} \leq (t - 1)\nu + |U_q| + |U_{q+1}|.$$  \hfill (29)

Finally, by (23), (28), and (29), and using $t \leq p$, we get

$$n = |V(C)| \leq |W| + (t - 1)\nu + |U_q| + |U_{q+1}|$$

$$\leq p(k + 2g) + (p - 1)\nu + |U_q| + |U_{q+1}|$$

$$< (q - 1)\alpha + |U_q| + |U_{q+1}| = n,$$

which is a contradiction. Hence, there is no $\ell$-overlapping Hamiltonian cycle in $H'$. \hfill $\Box$

Before we finalize our construction, we need one more piece of notation. For each $e \in \binom{V}{k}$ with $|tr(e)| \geq 2$, let

$$\min_2(e) = \min\{i : (e \setminus U_{\min(e)}) \cap U_i \neq \emptyset\}.$$  \hfill (30)

Finally, set

$$H_3 = \left\{ e \in \binom{V}{k} : |tr(e)| \geq 2 \quad \text{and} \quad \min_2(e) \geq q - 2k \right\},$$
Lemma 20. \( H' = H_1 \cup H_2 \cup H_3 \), and let \( H \) be a maximal non-\( \ell \)-Hamiltonian \( k \)-graph such that \( H' \subseteq H \subseteq H'' \). By Lemma 18, such a \( k \)-graph \( H \) exists.

**Fact 19.**
\[
|H| = O(n^{(k+\ell)/2})
\]

**Proof.** By the definitions of \( H \) and \( H'' \),
\[
|H| \leq |H''| \leq |H_1| + |H_2| + |H_3|.
\]

Now, noticing that \( \max_{1 \leq i \leq q+1} |U_i| = \beta \), we have
\[
|H_1| \leq \sum_{i=1}^{q+1} \binom{|U_i|}{k-\ell+1} \cdot \binom{n}{\ell-1} \leq (q+1) \cdot \beta^{k-\ell+1} \cdot n^{\ell-1} = O(n^{(k+\ell)/2}),
\]
\[
|H_2| \leq \left( \binom{|U_q|}{k-2} \cdot \binom{n}{2} \right) \leq \beta^{k-2} \cdot n^2 = O(n^{(k+2)/2}), \text{ and}
\]
\[
|H_3| \leq \sum_{i=1}^{q} \sum_{t=1}^{k-1} \binom{|U_i|}{t} \cdot \binom{|U_{q-2t}| + \cdots + |U_{q+t}|}{k-t} = O\left(q \cdot \alpha^t \cdot \beta^{k-t}\right) = O\left(n^{(k+1)/2}\right),
\]
where \( i = \min(e) \) and \( t = \min(e \cap U_{\min(e)}) \).

To complete the proof of Theorem 3, it remains to show the following lemma.

**Lemma 20.** For every \( e \in \binom{V}{k} \setminus H \) the \( k \)-graph \( H + e \) is \( \ell \)-Hamiltonian.

**Proof.** Fix \( e \in \binom{V}{k} \setminus H \). If \( e \in H'' \), then, by the definition of \( H \), \( H + e \) is \( \ell \)-Hamiltonian. Therefore, we may assume that \( e \notin H'' \). This implies that \( |tr(e)| \geq 2 \), since otherwise \( e \in H_1 \). Define
\[
x = \min(e) \quad \text{and} \quad y = \min_2(e).
\]

Since \( e \notin H_1 \cup H_3 \), we have \( |U_x \cap e| \leq k - \ell \) and \( x < y \leq q - 2k - 1 \).

Our ultimate goal is to construct in \( H \) an \( \ell \)-overlapping Hamiltonian cycle \( C \). Recalling (26), let \( J = \{j_1, \ldots, j_{p-2}\} \) be the set of the \( p - 2 \) smallest indices in the set \( \{1, \ldots, q - 2k - 1\} \setminus \{x, y\} \). Further, let
\[
r_i = |e \cap U_i|, \quad i = 1, \ldots, q + 1.
\]

Since \( e \notin H_2 \), we have \( r_{q+1} \leq k - 3 \). Thus \( |U_{q+1} \setminus e| \geq p(k - 2) \). Let us now set aside \( p \) disjoint \( (k - 2) \)-element subsets \( B_1, \ldots, B_p \) of \( U_{q+1} \setminus e \) and let
\[
B = \bigcup_{i=1}^{p} B_i.
\]

Note that
\[
|U_{q+1} \setminus (B \cup e)| = k - 3 - r_{q+1} \leq k.
\]

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Furthermore, let us also put aside a set $Q = A_q \cup A'_q$ of $2(g + 1)$ elements of $U_q \setminus e$, where $|A_q| = |A'_q| = g + 1$. The vertices in $B$ and $Q$ will be used later in our construction.

First, however, we construct $p$ vertex disjoint paths $P_1, \ldots, P_{p-2}$, $P_{xy}$ and $P_q$. Together, these $p$ paths will contain all elements of $V$, except for some $k - \ell + g + 1$ vertices of $U_x$, the same number of vertices of $U_y$, twice as many vertices of each $U_j$, $j \in J$, and except for the vertices in $B \cup Q$. Using these exceptional vertices, the paths will be connected by $p$ ‘bridges’, made mostly of the edges of $H_2$, to form an $\ell$-overlapping Hamiltonian cycle $C$ in $H$.

**Construction of $P_{xy}$**. Order the vertices of $e$ so that the set $e \cap U_x$ constitutes the leftmost segment of $e$, while the rightmost vertex of $e$ belongs to $U_y$. Next, we will extend $e$ in both directions (see Fig. 1). Let $A'_x$ be a set of arbitrary $k - \ell + g$ vertices of $U_x \setminus e$ and $A_y$ be a set of arbitrary $k - \ell + g$ vertices of $U_y \setminus e$ (the reader should not worry, we will later construct sets $A_x$ and $A'_y$ too). Let

$$R = \bigcup_{i=q-2k}^{q-1} U_i \setminus e.$$

Further, for each $z \in \{x, y\}$, let $P_z \subseteq C_z$ be a path containing precisely

$$\alpha_z := \alpha - r_z - (2k - 2\ell + 2g + 1)$$

vertices of $U_z \setminus (e \cup A'_x \cup A_y)$ and $\nu(\alpha_z) - \alpha_z$ vertices of $R$, where $V(P_x) \cap V(P_y) = \emptyset$. Since, by Proposition 7, each of $P_x$ and $P_y$ requires no more than $(k - 1)\alpha$ vertices of $R$, while $|R| \geq 2k\alpha - k$, we will not run out of the vertices of $R$.

To finish the construction of $P_{xy}$, we extend $e$

- to the left, by adding the set $A'_x$, followed by $P_x$, and
- to the right, by adding the set $A_y$, followed by $P_y$.

Thus,$$
V(P_{xy}) = V(P_x) \cup A'_x \cup e \cup A_y \cup V(P_y) \subseteq U_x \cup U_y \cup e \cup R.
$$

Set

$$A_x = U_x \setminus V(P_{xy}) \quad \text{and} \quad A'_y = U_y \setminus V(P_{xy})$$

and observe that

$$|A_x| = |A'_y| = k - \ell + g + 1.$$  \hspace{1cm} (32)

**Fact 21.**

$$P_{xy} \subseteq H_1 + e$$

**Proof.** The path $P_{xy}$ consists, besides the edges of $P_x$, $P_y$, and $e$ itself, also of a set $A$ of $2\left\lceil \frac{k}{k-\ell} \right\rceil$ additional edges, $\left\lfloor \frac{k}{k-\ell} \right\rfloor$ on each side of $e$. These are precisely those edges of $P_{xy}$ which intersect the set $A'_x \cup A_y$. Thus, to prove that $P_{xy} \subseteq H_1 + e$, it remains to show that each edge from $A$ belongs to $H_1$. 

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Let us consider an edge $e'$ intersecting $A'_x$. Obviously, $\min(e') = x$. Also, $|e' \cap A'_x| \geq k - \ell$, and so $|e' \cap U_x| \geq k - \ell$. Furthermore, if $|e' \cap A'_x| = k - \ell$ then either $e'$ contains also the leftmost vertex of $e$ (which belongs to $U_x$), or $|e' \cap V(P_x)| = \ell$. In the latter case, recall that each edge of $P_x$ contains at least $k - \ell + 1$ vertices from $U_x$, and consequently there is always a vertex form $U_x$ among any $\ell$ vertices of such an edge. In either case, this implies that $|e' \cap U_x| \geq k - \ell + 1$, and so $|e' \cap U_x| \geq k - \ell$. Furthermore, if $|e' \cap A'_x| = k - \ell$ then either $e'$ contains also the leftmost vertex of $e$ (which belongs to $U_x$), or $|e' \cap V(P_x)| = \ell$. In the latter case, recall that each edge of $P_x$ contains at least $k - \ell + 1$ vertices from $U_x$, and consequently there is always a vertex form $U_x$ among any $\ell$ vertices of such an edge. In either case, this implies that $|e' \cap U_x| \geq k - \ell + 1$, and so $|e' \cap U_x| \geq k - \ell$. Indeed, since $|U_x \cap e| \leq k - \ell$, none of the $\ell$ rightmost vertices of $e$ is in $U_x$, and hence, we have $e' \cap U_x = \emptyset$.

**Construction of $P_q$.** Let $P_q$ be a longest path with $V(P_q) \subset U_q \setminus (e \cup Q)$. Clearly, at most $k - \ell - 1$ vertices of $U_q$ will be left out, that is,

$$|U_q \setminus (V(P_q) \cup e \cup Q)| \leq k - \ell - 1 \leq k.$$  \hfill (33)

Trivially, $P_q \subset H_1$.

**Construction of $P_j$, $j \in J$.** Set

$$W := \left( \bigcup_{i \in \{1, \ldots, q+1\}\setminus(J\cup\{x,y\})} U_i \right) \setminus (V(P_{xy}) \cup V(P_q) \cup B \cup Q \cup e),$$

and, for each $j \in J$, let $P_j \subseteq C_j \subseteq H_1$ be a path with $V(P_j) \subseteq U_j \cup W$ which uses precisely

$$\alpha_j := \alpha - r_j - (2k - 2\ell + 2g + 2)$$

vertices of $U_j \setminus e$ and as many as possible vertices from $W$ (we maintain that all paths $P_j$, $j \in J$, are pairwise vertex-disjoint). Since $i > j$ for every $i \in [q+1] \setminus (J \cup \{x,y\})$, we do have $\min(V(P_j)) = j$. Also, set

$$|U_j \setminus (V(P_j) \cup e)| = 2(k - \ell + g + 1) \quad \text{for each } j \in J.$$  \hfill (34)

Split arbitrarily the set $U_j \setminus (V(P_j) \cup e)$ into two sets $A_q$ and $A'_q$ of equal size $|A_q| = |A'_q| = k - \ell + g + 1$.

Next, we perform crucial calculations showing that we have, indeed, used all the vertices of $W$, that is, there are no vertices outside the constructed paths except for those listed in (32,34) and those put aside in $B \cup Q$. 

**Figure 1:** Construction of $P_{xy}$
**Fact 22.**  

\[ W \subseteq \bigcup_{j \in J} V(P_j) \]

Proof. We have, by the definition of \( P_{xy} \), and by (31) and (33),

\[
|W| = (q - 1 - p)\alpha - |R \cap V(P_{xy})| + |U_q \setminus (V(P_q) \cup e \cup Q)| + |U_{q+1} \setminus (B \cup e)|,
\]

\[
\leq (q - 1 - p)\alpha - (\nu(\alpha_x) - \alpha_x) - (\nu(\alpha_y) - \alpha_y) + 2k.
\]

Recall that each path \( P_j, j \in J \), may have the maximum length \( \nu(\alpha_j) \), and thus cover up to \( \nu(\alpha_j) - \alpha_j \) vertices of \( W \). Therefore, to complete the proof it suffices to show that

\[
(q - 1 - p)\alpha - (\nu(\alpha_x) - \alpha_x) - (\nu(\alpha_y) - \alpha_y) + 2k \leq \sum_{j \in J} (\nu(\alpha_j) - \alpha_j),
\]

or, equivalently,

\[
\sum_{j \in J \cup \{x, y\}} (\nu(\alpha_j) - \alpha_j) \geq (q - 1 - p)\alpha + 2k.
\]

Note that for each \( j \in J \cup \{x, y\} \)

\[
r_j + 2k - 2\ell + 2g + 2 \leq 5k. \tag{35}
\]

Hence, by the monotonicity of the function \( \nu(\cdot) \) and by Proposition 9, we have

\[
\nu(\alpha_j) - \alpha_j \geq \nu(\alpha - 5k) - \alpha \geq \nu - 5k^2 - \alpha,
\]

and it remains to show that

\[
p(\nu - 5k^2 - \alpha) \geq (q - 1 - p)\alpha + 2k. \tag{36}
\]

To this end,

\[
p(\nu - 5k^2) - p\alpha \geq (p - 1)\nu + (\alpha + \alpha/(k - 1) - k) - 5k^2 p - p\alpha \quad \text{(by Corollary 10)}
\]

\[
\geq (p - 1)\nu + \alpha + p(k + 2g) + 2k - p\alpha \quad \text{(by (24))}
\]

\[
\geq (q - 1 - p)\alpha + 2k \quad \text{(by (23))}.
\]

(Since there is some margin in the above estimates, it means that not all the paths \( P_j, j \in J \), are of maximum length.)

Now comes the final stage of our construction, where we glue together the paths \( P_{j_1}, \ldots, P_{j_{p-2}}, P_q, \) and \( P_{xy} \), in this order, to form a Hamiltonian cycle \( C \). We do it as indicated in Fig. 4, with the set \( A_x \) placed at the left end of \( P_{xy} \), that is, next to the end of the path \( P_x \) (see Fig. 4).

Clearly, every edge of \( \bigcup_{i=1}^{p-2} P_i \cup P_{xy} \cup P_q \) belongs to \( H + e \). As the last ingredient of our proof of Theorem 3, we now show that every other edge of \( C \) belongs to \( H_1 \cup H_2 \subseteq H \).
Fact 23.

\[ C \setminus \left( \bigcup_{i=1}^{p-2} P_j \cup P_{xy} \cup P_q \right) \subseteq H_1 \cup H_2 \]

Proof. Let

\[ A := \{ A_{ji}, A'_{ji} : i = 1, \ldots, p - 2 \} \cup \{ A_q, A'_q, A_x, A'_y \}. \]

Note that each edge of \( C \setminus \left( \bigcup_{i=1}^{p-2} P_j \cup P_{xy} \cup P_q \right) \) intersects some set \( A \in A \). Recall that between any two disjoint edges of \( C \) there are exactly \( g + t(k - \ell) \) vertices on \( C \), for some \( t \geq 0 \). In that case we say that the edge to the right (in some fixed ordering of \( C \)) \( t \)-follows the other edge. Let \( f_1 \), be the edge of \( C \) which \( 1 \)-follows the rightmost edge of \( P_{xy} \). Similarly, for \( i = 1, \ldots, p - 2 \), let \( f_{i+1} \) be the edge of \( C \) which \( 1 \)-follows the rightmost edge of \( P_{ji} \). Finally, let \( f_p \) be the edge of \( C \) which \( 1 \)-follows the rightmost edge of \( P_q \), see Fig. 4. Note that for each \( i = 1, \ldots, p \), we have \( B_i \subset f_i \), and thus \( f_i \in H_2 \). Furthermore, these are the only edges of \( C \) which intersect more than one set from \( A \).

Consider now some \( f \in C \), \( f \neq f_i \) intersecting \( A_{ji} \). Obviously \( \min(f) = j_i \). Also \( |f \cap A_{ji}| \geq k - \ell \). However, if \( |f \cap A_{ji}| = k - \ell \), then \( |f \cap V(P_{ji})| = \ell \). Recall that each edge of \( P_{ji} \) contains at least \( k - \ell + 1 \) vertices of \( U_{ji} \), and consequently there is always a vertex of \( U_{ji} \) among any \( \ell \) vertices of such an edge. This implies that \( |f \cap U_{ji}| \geq k - \ell + 1 \) and so, \( f \in H_1 \). The same argument works for any \( f \in C \) intersecting some set \( A \in A \). \( \square \)

Thus, we have constructed an \( \ell \)-overlapping Hamiltonian cycle \( C \) in \( H + e \), which completes the proof of Lemma 20, which together with Fact 19, implies Theorem 3.

5 The smallest open case: \( k = 4 \) and \( \ell = 2 \)

In this section we prove Theorem 4. Our ultimate goal is, given large even integer \( n \), to construct a maximally non-2-Hamiltonian 4-graph \( H \). In doing so we refine the technique used in the proof of Theorem 3.

Choose integers \( \alpha = \Theta(n^{2/5}) \), \( \alpha \equiv 1 \mod 3 \), \( \beta = O(n^{3/5}) \), \( p = \Theta(n^{3/5}) \), and

\[ q = \left\lfloor \frac{4(\alpha - 1)}{3\alpha} (p - 1) \right\rfloor + 1 \tag{37} \]

such that

\[ n = q\alpha + 3p + \beta. \tag{38} \]
To see that such a choice is feasible, one may set, for instance, $\alpha = \left\lceil \frac{n^2}{5} \right\rceil + \epsilon$ where $\epsilon \in \{0, 1, 2\}$ is such that $\alpha \equiv 1 \mod 3$. Next choose $p = \left\lceil \frac{3n}{4\alpha + 8} \right\rceil + 1$. Then, using (37,38) we have

\[
\begin{align*}
    n - \beta &> \frac{4}{3} (\alpha - 1)(p - 1) \geq n - \frac{3n}{\alpha + 3} \\
    n - \beta &\leq \frac{4}{3} (\alpha - 1)(p - 1) + \alpha + 3p = (p - 2) \left( \frac{4}{3} (\alpha - 1) + 4 \right) - \left( p - \frac{7}{3} (\alpha - 1) - 9 \right)
\end{align*}
\]

which shows that a choice of an appropriate $\beta$ is possible.

Let $V = \bigcup_{i=1}^{q+1} U_i$, where $|U_i| = \alpha$, $i = 1, \ldots, q$, while $|U_{q+1}| = 3p + \beta$, and all sets $U_i$, $i = 1, \ldots, q + 1$, are pairwise disjoint. Furthermore, let $G \cong pK_3 + \beta K_1$ be a graph with vertex set $V(G) = U_{q+1}$ consisting of $p$ vertex disjoint triangles and $\beta$ isolated vertices.

We define $H_1$ in the same way as in the general case, while $H_2$ is defined smaller:

\[
\begin{align*}
    H_1 &= \left\{ e \in \binom{V}{4} : |e \cap U_{\min(V)}| \geq 3 \right\}, \\
    H_2 &= \left\{ e \in \binom{V}{4} : |e \cap U_{q+1}| = 2, |tr(e)| = 2 \text{ and } G[e \cap U_{q+1}] = K_2 \right\}. \tag{39}
\end{align*}
\]

The improvement of the upper bound on $\text{sat}(n, 4, 2)$ is possible mainly because in this particular case one can compute (quite easily) the value of $\nu(x)$. Below we give only a (sharp) upper bound in some special case.

**Proposition 24.** Let $x \equiv 0 \mod 3$. Then

$$
\nu(x) \leq 4 \frac{x}{3}.
$$

**Proof.** Let $P = (e_1, \ldots, e_r)$, $P \subseteq H_1$ and $|V(P) \cap U_{\min(V(P))}| = x$. Recall that each $e_i$, $i = 1, \ldots, r$, contains at least 3 vertices from $U_{\min(V(P))}$. Since the $e_i$'s with odd indices are disjoint,

$$
[r/2] \leq \frac{x}{3}.
$$

If $r$ is odd then

$$
|V(P)| \leq 4 \lfloor r/2 \rfloor \leq 4 \frac{x}{3}
$$

and the statement follows. Similarly, if $r$ is even and $r/2 \leq \frac{x}{3} - 1$ then

$$
|V(P)| \leq 2r + 2 \leq 4 \frac{x}{3} - 2
$$

and the statement follows again. Suppose, finally, that $r/2 = \frac{x}{3}$, $r$ even. Since $e_r$ contains at least 3 vertices from $U_{\min(V(P))}$, at least one of them is not in $e_{r-1}$, however there are no more available vertices in $U_{\min(V(P))}$, meaning that this case is vacuous. □
Lemma 25. $H' = H_1 \cup H_2$ is not 2-Hamiltonian.

Proof. Suppose that $C$ is a 2-overlapping Hamiltonian cycle in $H'$. As before (cf. Corollary 17), one can easily show that $H_1$ cannot be 2-Hamiltonian. Let $M$ be a maximal set of edges in $C \cap H_2$ with the property that if $e_1, e_2 \in M$ then $(e_1 \cap e_2) \cap U_{q+1} = \emptyset$. In view of the above remark $M \neq \emptyset$. Set

$$V_2 = \bigcup_{e \in M} e \cap U_{q+1}.$$ 

Clearly, $t := |M| \leq p$ and $|V_2| = 2t$. We divide $C$ into $t$ vertex disjoint paths $P_j$, $j = 1, \ldots, t$, by cutting through the middle of every edge from $M$ (we treat a set of 2 consecutive isolated vertices as a single trivial path). More precisely, we keep all vertices in and take the edge set $C - M$. We number the obtained paths so that, for some $1 \leq s \leq t$, we have $\min(V(P_j)) \leq q$ for all $j = 1, \ldots, s$ and $V(P_j) \subseteq U_{q+1}$ for all $j = s + 1, \ldots, t$. Note that, because $M \neq \emptyset$, at least one path must be of the first kind, but possibly $s = t$. Let

$$V_2' = V_2 \cap \bigcup_{j=1}^{s} V(P_j).$$

Since $V(P_j) \subseteq U_{q+1}$ for all $j = s + 1, \ldots, t$, we have

$$\sum_{j=s+1}^{t} |V(P_j)| \leq |U_{q+1}| - |V_2'|. \quad (40)$$

Claim For every $j = 1, \ldots, s$

$$|V(P_j) \setminus V_2'| \leq 4\frac{\alpha - 1}{3}. \quad (41)$$

Proof. If some $P_j$ consists of only two vertices then the claim obviously holds. Thus, we may assume that each $P_j$ is non-trivial. For $j \leq s$, consider the path $P_j = (e_1, \ldots, e_r)$. Let $e_m \in M$ with $|e_m \cap e_1| = 2$. That is $e_m$ precedes $e_1$ on $C$. Similarly, let $e_{r+1} \in M$ with $|e_{r+1} \cap e_r| = 2$, which means that $e_{r+1}$ follows $e_r$ on $C$.

Note that the edges from $H_2$ can occur in $P_j$ only at the ends. Thus $(e_2, \ldots, e_{r-1}) =: P_j' \subseteq H_2$. If $e_1 \in H_1$ then $|e_1 \cap U_{\min(V(P_j))}| \geq 3$, meaning that $|e_m \cap U_{\min(V(P_j))}| \geq 1$. Thus, by the definition of $H_2$, $|e_m \cap U_{\min(V(P_j))}| = 2$. If $e_1 \in H_2$ then, since $e_1 \notin M$, we have $|e_1 \cap V_2'| \in \{1, 2\}$. If $|e_1 \cap V_2'| = 1$ then $|e_m \cap U_{\min(V(P_j))}| \geq 1$ because $|e_m \cap e_1| = 2$ and $|tr(e_1)| = 2$. Thus, again, $|e_m \cap U_{\min(V(P_j))}| = 2$. To sum up

$$\text{if } e_1 \in H_1 \text{ or } |e_1 \cap V_2'| = 1 \text{ then } |e_m \cap U_{\min(V(P_j))}| = 2. \quad (41)$$

The same holds for $e_r$ and $e_{r+1}$

$$\text{if } e_r \in H_1 \text{ or } |e_r \cap V_2'| = 1 \text{ then } |e_{r+1} \cap U_{\min(V(P_j))}| = 2. \quad (42)$$
Suppose first that the assumptions on both \( e_1 \) and \( e_r \) from (41,42), respectively, holds. Thus, \( |V(P'_j) \cap U_{\min(V(P_j))}| \leq \alpha - 4 \). Since \( \alpha - 4 \equiv 0 \mod 3 \), by Proposition 24 and the monotonicity of the function \( \nu \),

\[
|V(P_j)| = |V(P'_j)| + 4 \leq 4 \frac{\alpha - 4}{3} + 4 = 4 \frac{\alpha - 1}{3}
\]

and the claim follows.

Suppose now that \( e_1 \in H_2 \) with \( |e_1 \cap V'_2| = 2 \), while \( e_r \) satisfies the assumptions from (42). Let \( P''_j \) be defined by \( (e_3, \ldots, e_{r-1}) \). By the definition of \( H_2 \), \( |e_1 \cap U_{\min(V(P_j))}| = 2 \). This together with (42) implies that \( |V(P''_j) \cap U_{\min(V(P_j))}| \leq \alpha - 4 \). Hence, by Proposition 24 and the assumption on \( e_1 \),

\[
|V(P_j) \setminus V'_2| = (|V(P'_j)| + 6) - 2 \leq 4 \frac{\alpha - 4}{3} + 4 = 4 \frac{\alpha - 1}{3}
\]

and the claim follows again.

The case when \( e_1 \) satisfies the assumption of (41) and \( |e_r \cap V'_2| = 2 \), is analogous (with \( P''_j = (e_2, \ldots, e_{r-2}) \)).

Finally, if \( |e_1 \cap V'_2| = 2 \) and \( |e_r \cap V'_2| = 2 \) then let \( P''_j = (e_3, \ldots, e_{r-2}) \). Since \( e_1, e_r \in H_2 \) (and \( e_2, e_{r-1} \in H_1 \)), we have \( |e_1 \cap U_{\min(V(P_j))}| = 2 \) and \( |e_r \cap U_{\min(V(P_j))}| = 2 \). Therefore,

\[
|V(P_j) \setminus V'_2| = (|V(P''_j)| + 8) - 4 \leq 4 \frac{\alpha - 4}{3} + 4 = 4 \frac{\alpha - 1}{3}
\]

and the claim follows.

Returning to the proof of Lemma 25, notice that \( |V'_2| \leq |V_2| = 2t \leq 2p \). Thus

\[
|U_{q+1}| = 3p > |V'_2| + 4 \frac{\alpha - 1}{3},
\]

because \( p >> \alpha \). Recalling that \( q > \frac{4(\alpha - 1)}{3\alpha} (p - 1) \) and using the above claim as well as (40,43), we finally argue that

\[
n = |V(C_H)| = \sum_{j=1}^{s} |V(P_j)| + \sum_{j=s+1}^{t} |V(P_j)| \leq \max\{|V'_2| + 4t \frac{\alpha - 1}{3}, |V'_2| + 4(t - 1) \frac{\alpha - 1}{3} + |U_{q+1}| - |V'_2|\},
\]

(according to wheather \( s = t \) or \( s \leq t - 1 \))

\[
= |V'_2| + 4(t - 1) \frac{\alpha - 1}{3} + |U_{q+1}| - |V'_2| \quad \text{by (43)}
\]

\[
\leq 4(p - 1) \frac{\alpha - 1}{3} + 3p < qa + 3p \leq n,
\]

which is a contradiction. Hence, no 2-overlapping Hamiltonian cycle exists in \( H_1 \cup H_2 \).
Let
\[ H_3 = \left\{ e \in \binom{V}{4} : |\text{tr}(e)| \geq 2 \text{ and } \min_2(e) \geq q \right\} \]
be the same as in the proof of Theorem 3. Finally, let \( H'' = H_1 \cup H_2 \cup H_3 \) and let \( H \) be a maximal non-2-Hamiltonian hypergraph such that \( H' \subset H \subset H'' \). By Lemma 25, such a 4-graph exists.

**Fact 26.**
\[ |H| = O(n^{14/5}) \]

**Proof.** By the definitions of \( H \) and \( H'' \),
\[ |H| \leq |H''| \leq |H_1| + |H_2| + |H_3|. \]
Furthermore,
\[ |H_1| = O\left(q \cdot \alpha^3 \cdot n + p^4\right) = O\left(n^{14/5}\right), \]
\[ |H_2| = O\left(3p \cdot n \cdot n^{2/5}\right) = O\left(n^2\right) \text{ and} \]
\[ |H_3| = O\left(n \cdot p^3\right) = O\left(n^{14/5}\right). \]

To complete the proof of Theorem 4, it remains to show the following lemma.

**Lemma 27.** For every \( e \in \binom{V}{4} \setminus H \) the 4-graph \( H + e \) is 2-Hamiltonian.

**Proof.** Let \( e = \{u_1, u_2, u_3, u_4\} \), where \( u_j \in U_j, j = 1, 2, 3, 4 \), and \( i_1 \leq i_2 \leq i_3 \leq i_4 \). As \( e \notin H_1 \), we have \( |\text{tr}(e)| \geq 2 \). Let \( x \) and \( y \) stand for the two smallest different indices among \( i_1, i_2, i_3, i_4 \). Note that by the definition of \( H, e \notin H_3 \), and thus \( y \leq q - 1 \).

Set \( I = [q - 1] \setminus \{x, y\} \), note that \( p - 2 \) is (much) smaller than \( q - 3 \), and let \( J = \{j_1, \ldots, j_{p-2}\} \) be the set of the \( p - 2 \) smallest indices in \( I \). We will construct \( p \) paths \( P_{j_1}, \ldots, P_{j_{p-2}}, P_{xy} \), and \( P_{q+1} \), such that for each \( j \in J \), we have \( V(P_j) \supseteq U_j \setminus e, \)
\[ U_x \cup U_y \cup e \subseteq V(P_{xy}) \subset U_x \cup U_y \cup e \cup U_q, \]
and \( V(P_{q+1}) \subset U_{q+1} \). Together, these paths will contain all vertices in \( V \) except some \( 2p \) vertices of \( U_{q+1} \). Using these exceptional vertices, the paths will be connected by \( p \) 'bridges' made of the edges of \( H_2 \), to form a 2-Hamiltonian cycle in \( H \).

For the ease of notation assume that \( x = q - 2 \) and \( y = q - 1 \). Then \( J = [p - 2] \).

To display the structure of each path we will use a shorthand notation \( j \) for any element of \( U_j \), \( j = 1, \ldots, p - 2, x, y, q, q + 1 \). Finally, we designate by * each of the two unknown elements of \( e = \{u_1, u_2, u_3, u_4\} \) (other than \( x \) and \( y \)); recall that \( u_1 \in U_x \), while \( \{u_2, u_3, u_4\} \subseteq \bigcup_{i=x}^{q+1} U_i \) and \( |\{u_2, u_3, u_4\} \cap U_x| \leq 1 \).

**Construction of \( P_{xy} \).** We consider five cases with respect to the multiplicities of the vertices of \( V_x \) and \( V_y \) in \( e \).
Case 1. In the case when $u_1 \in U_x$, $u_2 \in U_y$ and none of $u_3, u_4$ belongs to $U_y$, the path $P_{xy}$ is constructed as follows:

$$xx|xx|xx|qx|xx|qx|xx|\ldots|qx|xx|xx|*|yy|yy|yy|yy|yy$$

(the sequence begins with 3 blocks $|xx|$ followed by $(\alpha - 7)/3$ pairs $|qx|xx|$ and the edge $e$; the right side is constructed similarly with $y$ replacing $x$ and the blocks being arranged in the opposite order), where every element of $U_x \cup U_y$ appears exactly once, while $\frac{2}{3}(\alpha - 7) \leq |V(P_{xy}) \cap U_q| \leq \frac{2}{3}(\alpha - 7) + 2$ or equivalently $\frac{2}{3}(\alpha - 1) - 4 \leq |V(P_{xy}) \cap U_q| \leq \frac{2}{3}(\alpha - 1) - 2$ (recall that $3|\alpha - 1|$). Note that each pair of consecutive blocks of size two forms an edge of $H_1$ (except the middle pair $x*|*y$, which is just the edge $e$) and $|V(P_{xy})| = 2\left(4\frac{\alpha - 7}{3} + 8\right) = \frac{8}{3}(\alpha - 1)$.

Case 2. If $u_1 \in U_x$, $u_2 \in U_y$ and exactly one of $u_3, u_4$ belongs to $U_y$, the path $P_{xy}$ is constructed as follows:

$$xx|xx|xx|qx|xx|\ldots|qx|xx|xx|*|yy|yy|yy|yy|yy|yy$$

Again, $|V(P_{xy})| = \frac{8}{3}(\alpha - 1)$, while $\frac{2}{3}(\alpha - 1) - 3 \leq |V(P_{xy}) \cap U_q| \leq \frac{2}{3}(\alpha - 1) - 2$.

Case 3. If $u_1 \in U_x$ and $u_2, u_3, u_4 \in U_y$ then we form $P_{xy}$ as follows:

$$xx|xx|xx|qx|xx|\ldots|qx|xx|xx|xy|yy|yy|yy|yy|yy|yy$$

This time $|V(P_{xy})| = \frac{8}{3}(\alpha - 1) - 2$ and $|V(P_{xy}) \cap U_q| = \frac{2}{3}(\alpha - 1) - 4$.

Case 4. If $u_1, u_2 \in U_x$, $u_3 \in U_y$ and $u_4 \notin U_y$, the path $P_{xy}$ is constructed as follows:

$$xx|xx|qx|xx|\ldots|qx|xx|xx|*|yy|yy|yy|yy|yy|yy$$

Now $|V(P_{xy})| = \frac{5}{3}(\alpha - 1)$ and $\frac{2}{3}(\alpha - 1) - 3 \leq |V(P_{xy}) \cap U_q| \leq \frac{2}{3}(\alpha - 1) - 2$.

Case 5. If $u_1, u_2 \in U_x$ and $u_3, u_4 \in U_y$, we form the path $P_{xy}$ as follows:

$$xx|xx|qx|xx|\ldots|qx|xx|xx|*|yy|yy|yy|yy|yy|yy$$

We have again $|V(P_{xy})| = \frac{8}{3}(\alpha - 1)$, while $|V(P_{xy}) \cap U_q| = \frac{2}{3}(\alpha - 1) - 2$.

Let us now set aside $p$ 2-element disjoint subsets $B_1, \ldots, B_p$ of $U_{q+1}$ which correspond to disjoint edges of the graph $G$, one from each triangle of $G$. Set $B = \bigcup_{i=1}^{p} B_i$. These pairs will be used to glue together all $p$ paths into a Hamiltonian 2-cycle.

To describe the remaining paths, let symbol $w$ represent any element of the set

$$W := \bigcup_{i=p-1}^{q-3} U_i \cup U_q \cup (U_{q+1} \setminus B) \setminus V(P_{xy}).$$
Construction of \( P_j \), \( j = 1, \ldots, p-2 \). For \( j = 1, \ldots, p-2 \), we build path \( P_j \) by splitting \( \alpha - 4 \) vertices of \( U_j \) into \( (\alpha - 4)/3 \) blocks of length 3, separating them by arbitrary vertices from \( W \) and putting the remaining 4 vertices of \( U_j \) at the end. In a diagram form

\[
P_j = jj|jw|jj|jw| \ldots |jj|jw|jj|jj.
\]

Because \( j < \min\{i : U_i \cap W \neq \emptyset \} \), each pair of consecutive blocks of size two forms an edge of \( H_1 \). Also, \( |V(P_j)| = \frac{4}{3}(\alpha - 1) \), which means that \( P_j \) can accommodate precisely \( (\alpha - 4)/3 \) vertices from \( W \). As, by our choice of \( q \),

\[
(p-2)\frac{\alpha-4}{3} \geq (q-p-1)(\alpha-1) + \frac{\alpha-1}{3} + 3, \tag{44}
\]

we have

\[
\bigcup_{r=1}^{p-2} V(P_j) \supseteq \bigcup_{i=p-1}^{q-3} U_i \cup (U_q \setminus V(P_{xy})).
\]

On the other hand, the difference between the L-H-S and R-H-S of (44) is less than \( 42\alpha \) \( \ll p \), so that the surplus \( \omega \)-spots can be filled with some elements of \( U_{q+1} \).

Construction of \( P_{q+1} \). The last path, \( P_{q+1} \), consists of all the remaining vertices of \( U_{q+1} \) whose number is even, because \( n \) is even and every so far built path, as well as the set \( B \), consists of an even number of vertices.

The constructed paths \( P_1, \ldots, P_{p-2}, P_{xy} \), and \( P_{q+1} \) are now connected together, in arbitrary order, by the 2-element blocks \( B_1, \ldots, B_p \). Note that each \( B_j \) makes edges of \( H_2 \) with arbitrary 2-element sets from some \( U_i \), \( i = 1, \ldots, q \). This completes the construction of a 2-Hamiltonian cycle in \( H + e \).

The proof of Theorem 4 follows immediately from Lemma 27 and Fact 26.

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References


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References


